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ON THE MINIMIZATION OF AIRPLANE RESPONSES TO RANDOM GUSTS

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SUMMARY

A theoretical study is made of the motions experienced by aircraft in reponse to sharp-edge, harmonic, and random gusts. For the sharp-edge and harmonic gusts, exact responses in normal acceleration and pitching velocity are presented for the rectangular wing flying at Mach number 1.2. These are compared with approximate solutions based on commonly used assumptions, and the validity of each of the assumptions is assessed. It is determined that the use of stability derivatives in place of indicial functions in the equations of motion does not significantly impair the accuracy of solutions for transient and harmonic response.

The problem of alleviating the airplane's response to random gusts is cast in a form amenable to treatment by the Wiener optimum filter theory. A derivation is given of the theoretical requirements of a compensating-force system that minimizes a linear combination of the airplane's mean-square normal acceleration and mean-square pitching velocity. Results of computations are presented which indicate the system may be successful in causing significant reductions of both motions.

INTRODUCTION

For many years, aerodynamicists have studied the motions and loads which aircraft experience when they encounter vertical gusts in their line of flight. The reason for this continued interest is readily understood when it is recalled that the stresses imposed by gusts may be among the most severe that an aircraft structure is required to withstand. The gust loading condition is therefore generally considered the crucial factor in determining the adequacy of a structural design. Furthermore, since the structural design plays a large role in determining the aircraft's weight, the influence of the gust loading on the structural design in turn serves as a limiting factor on the maximum range and speed of the vehicle. The slightly different viewpoints expressed in the two reasons have inspired two fields of research, corresponding roughly to the categories of analysis and synthesis. In the first, the necessity of including the gust loading condition in structural calculations led to the development of analytical methods by which the motions and loads imposed by specified gusts could be predicted. Second, the possibility of saving weight by lightening the structure inspired efforts to develop aerodynamic and electromechanical devices which operate to offset the imposed loads caused by gusts. A brief sketch follows outlining some of the major developments in these fields.

The earliest work in analysis is that of Wilson (ref. 1) who followed the classical stability theory of Bryan (ref. 2) and Bairstow, Jones, and Thompson (ref. 3). In this theory it is assumed that the transient behavior of the aerodynamic forces and moments following sudden changes in airplane motion or gust velocity can be neglected. As a result, the equations of motion describing the airplane's response to disturbances reduce to a system of ordinary differential equations which are generally readily solved. Because of its simplicity and adaptability, this theory has been the one most often used in subsequent analyses. Later theoretical developments include the introduction of the transient lift functions (refs. 4 and 5) and their use in connection with the operational calculus to describe more precisely the motions and loads caused by gusts of arbitrarily specified structure (refs. 6 and 7). The analysis in this case is exact within the framework of the theory used to derive the transient lift functions, but complete solutions generally involve the inversion of integral equations, which often require a forbidding amount of labor. A further limitation in either of the methods just described is the necessity of specifying mathematically the structure of the spacewise distribution of gust velocity through which the airplane flies. It is, therefore, not possible to handle with these methods the more realistic situation in which the gusts are distributed randomly in the airplane's path. A significant recent development which overcomes this limitation concerns the introduction of concepts derived from the field of statistical dynamics. Here, under the assumption that the random process is stationary and Gaussian, it is sufficient to characterize the random distribution of gusty air by a single statistical quantity, the correlation function. It is then possible to predict quantities like peak load factor, or maximum accelerations on a probability basis (refs. 8, 9, and 10). The statistical methods still require as fundamental quantities, however, the responses of the airplane to step or sinusoidal gust inputs, so that rather than being superseded, the analyses described previously for deriving these quantities take on added importance. For successful application, these methods also require a depiction of the statistical nature of atmospheric turbulence, and much effort is currently being devoted to this task (refs. 11, 12, and 13).

In the field of synthesis, a wide variety of proposals has been advanced for alleviating the effects of gusts. These include the use of spoilers (ref. 14), gaps in the wing (ref. 15), and the wing's aeroelastic properties (ref. 16) to destroy or offset the increments of lift caused by the gust velocity. Perhaps the most successful study has been that of Phillips and Kraft (ref. 17). These authors adapt the classical stability analysis of Wilson (ref. 1) and specify that the gust velocity disturbance be represented by a continuous harmonic function. The resulting mathematical simplifications enable them to study a variety of flap and elevator arrangements suitable for offsetting not only the increments in gust lift but pitching moment as well. Their results indicate that with proper selection of gearing and phase between elevator and flap deflections it is possible to reduce significantly the airplane's response in both normal acceleration and pitching velocity to the harmonic gust input. It remains for further investigations to determine whether inclusion of the transient lift functions in the equations of motion or a different portrayal of the gust inputs would result in appreciably different conclusions.

The present paper has two purposes, the first pertaining to analysis, the second to synthesis. First, in view of the widespread use of simplifying assumptions and approximations in gust response analyses, we consider the most prevalent of these in the light of more precise solutions. Accurate numerical results are presented for the responses in normal acceleration and pitching velocity to step and harmonic gust inputs for a rectangular wing flying at near-sonic speed. These results are compared with results containing the commonly used approximations and the validity of each approximation is assessed. Second, we wish to adapt to the gust alleviation problem a powerful synthetic development in statistical dynamics, not yet as widely used by aerodynamicists as the statistical methods of analysis previously cited. This is the optimum filter theory of Wiener (ref. 18). Wiener showed that if certain statistical information is given about the character of message and random noise inputs to a linear system, one can design a filter which acts to suppress the noise so that a mean-square error, specified to be a measure of the failure of the system to follow the message, is minimized. The analogous terms in the gust alleviation problem are obvious: The noise corresponds to the random distribution of gusty air, the message to the desired path of the airplane, and the filter to the characteristics of a controlling device which operates to minimize the airplane's response to the gusty air. Herein, the Wiener theory is applied to the gust alleviation problem to derive the transfer function of a control system which minimizes a linear combination of the airplane's mean-square normal acceleration and mean-square pitching velocity. A triangular wing flying at a high subsonic Mach number is used as an example in numerical calculations to assess the effectiveness of the control system in reducing the airplane's response to a specified statistical distribution of atmospheric turbulence.

NOTATION

A aspect ratio

C_L lift coefficient, $\frac{\text{lift}}{q_\infty S}$

C_m pitching-moment coefficient, $\frac{\text{pitching moment}}{q_\infty Sc}$

I	airplane pitching moment of inertia
I.P.	imaginary part
M	Mach number
R.P.	real part
S	wing area
V	flight speed
W	gust autocorrelation function (eq. (38))
a	weighting parameter in definition of mean-square error (eq. (33))
c	wing root chord
e	base of natural logarithms
k	stiffness factor (eq. (15))
i	$\sqrt{-1}$
m	airplane mass
q	dimensionless pitching velocity, $\frac{\dot{\theta}c}{V}$
q_∞	dynamic pressure, $\frac{1}{2}\rho_\infty V^2$
s	variable of Laplace transforms
t	time
w_g	gust vertical velocity
x,y,z	Cartesian coordinates
x_a	distance in root-chord lengths of aerodynamic center from center of gravity
x_o	distance in root-chord lengths of control force from center of gravity
α	angle of attack
α_g	$\frac{w_g}{V}$
β	$\sqrt{M^2-1}$

γ flight path angle

$\overline{\epsilon^2}$ mean-square error

ζ dimensionless inertia parameter, $\dfrac{IV^2}{q_\infty Sc^3}$

η dimensionless mass parameter, $\dfrac{mV^2}{q_\infty Sc}$

θ angle of pitch

λ reduced frequency, $\dfrac{c}{V} \times$ (angular frequency)

μ damping factor (eq. (15))

ρ_∞ mass density of free stream

φ chord lengths of travel in time t, $\dfrac{Vt}{c}$

Superscripts

$(\ )'$ $\dfrac{d}{d\varphi}\left(\ \right)$

$(\dot{\ })$ $\dfrac{d}{dt}\left(\ \right)$

$(\ )^*$ complex conjugate

$\overline{(\ )}$ mean value, defined as $\overline{f(\varphi)} = \lim\limits_{\Phi \to \infty} \dfrac{1}{2\Phi} \displaystyle\int_{-\Phi}^{\Phi} f(\varphi)\,d\varphi$

When α, $\dot{\alpha}$, and q are used as subscripts a dimensionless derivative is indicated, and this derivative is evaluated as the independent variable approaches zero. Thus

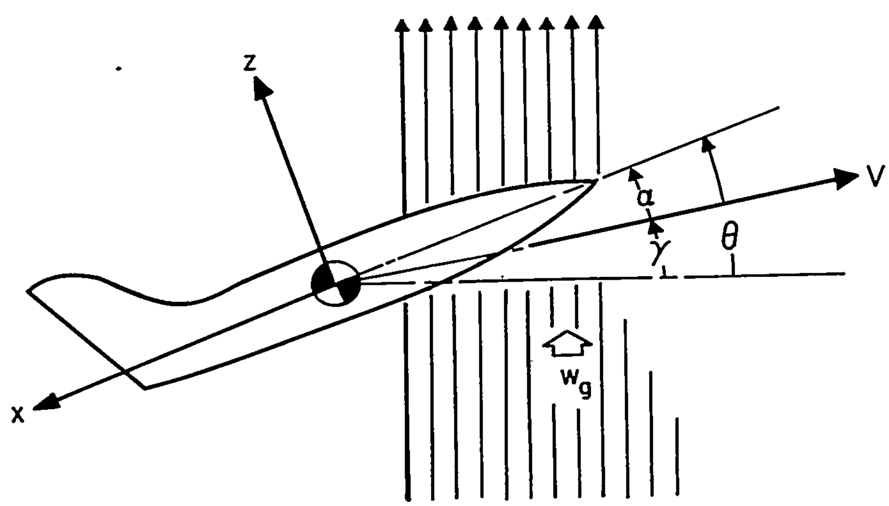
$$C_{m_\alpha} = \left(\frac{\partial C_m}{\partial \alpha}\right)_{\alpha \to 0}, \qquad C_{m_{\dot{\alpha}}} = \left(\frac{\partial C_m}{\partial \frac{\dot{\alpha} c}{V}}\right)_{\dot{\alpha} \to 0}, \qquad C_{m_q} = \left(\frac{\partial C_m}{\partial q}\right)_{q \to 0}$$

ANALYSIS

Airplane Motions in Response to Sharp-Edge Gust

Consider an airplane which moves forward in level steady flight and suddenly encounters a uniform field of vertical velocity. The boundary of the field is oriented parallel to the airplane's lateral axis and extends on either side beyond the widest span of the airplane. This is, of course, the mathematical idealization of the sharp-edge gust. As the airplane penetrates the gust front, loading develops on that portion of its surface influenced by the gust. The loading may be resolved into time-varying force and moment excitations which disturb the airplane's equilibrium and cause it to undergo a plunging and pitching motion. Our purpose will be to calculate this motion as a function of chord lengths of travel φ, letting $\varphi = 0$ correspond to the instant when the foremost point of the airplane just penetrates the gust front.

Coordinate system.- In sketch (a) is shown the position of the airplane at an instant when only a portion of the airplane has penetrated



Sketch (a)

the gust front. As shown in the sketch, an x,y,z coordinate system is placed in the airplane, the origin being fixed at the center of gravity. The instantaneous direction of the path of the center of gravity relative to still air is indicated by the direction of the flight velocity vector V; we assume that the magnitude of V remains constant throughout the motion. The positive branch of the x axis is rearward and alined with the chord line of the wing. The y axis is perpendicular to the vertical plane of symmetry and coincides with the lateral axis running through the center of gravity; the z axis lies in the vertical plane of symmetry, perpendicular to the xy plane. We define the angle of attack α to be the angle between the xy plane and the plane containing the velocity vector; the angle of pitch θ is the angle between the xy plane and the horizontal plane. The flight path angle γ is the angle between the horizontal and the plane containing the velocity vector; it is equal to $\theta-\alpha$. Finally, the gust velocity is indicated by w_g, and is measured relative to still air. All quantities are indicated in sketch (a) in their positive directions. Forces are measured positive upward, whereas pitching moments are positive when tending to increase the angle of pitch.

Equations of motion.- In this analysis, the indicial function concept will be used to write the equations which define the airplane motion. Since this will involve the use of both the superposition principle and the results of linearized unsteady-flow theory, the applicability of the analysis will be limited by the conditions under which their use can be justified. Hence, for example, disturbances and angular displacements must be small. It follows that the equations of motion may be written as

$$\left.\begin{aligned} \frac{mV^2}{c}\gamma'(\varphi) &= \Sigma \text{ forces} \\ I\left(\frac{V}{c}\right)^2 q'(\varphi) &= \Sigma \text{ pitching moments} \end{aligned}\right\} \tag{1}$$

where the independent variable is chord lengths of travel φ. The forces and moments to be summed are of three types, those caused (1) by variations of the angle of attack, $\alpha(\varphi)$; (2) by variations of pitching velocity, $q(\varphi)$; and (3) by the gust velocity, w_g. If the indicial lift and moment responses corresponding to step changes in each of the variables $\alpha(\varphi)$, $q(\varphi)$, and w_g are given, the forces and moments due to variations in these quantities may be built up by use of the superposition integral (cf., for example, ref. 19). For the case w_g = constant, equations (1) may be written as

$$\left.\begin{aligned} \eta\gamma'(\varphi) &= \frac{d}{d\varphi}\int_0^\varphi C_{L_\alpha}(\xi)\alpha(\varphi-\xi)d\xi + \frac{d}{d\varphi}\int_0^\varphi C_{L_q}(\xi)q(\varphi-\xi)d\xi + 1\alpha_g C_{L_g}(\varphi) \\ \zeta q'(\varphi) &= \frac{d}{d\varphi}\int_0^\varphi C_{m_\alpha}(\xi)\alpha(\varphi-\xi)d\xi + \frac{d}{d\varphi}\int_0^\varphi C_{m_q}(\xi)q(\varphi-\xi)d\xi + 1\alpha_g C_{m_g}(\varphi) \end{aligned}\right\} \tag{2}$$

where

$C_{L_\alpha}(\varphi)$, $C_{m_\alpha}(\varphi)$ indicial lift and pitching-moment responses to unit step change in angle of attack, α

$C_{L_q}(\varphi)$, $C_{m_q}(\varphi)$ indicial lift and pitching-moment responses to unit step change in dimensionless pitching velocity, $q = \frac{\dot{\theta}c}{V}$

$C_{L_g}(\varphi)$, $C_{m_g}(\varphi)$ indicial lift and pitching-moment responses to unit step change in dimensionless gust velocity

$1\alpha_g$ step change in dimensionless gust velocity, $\frac{w_g}{V}$

Equations (2) are a pair of integral equations involving the unknown variations $\alpha(\varphi)$, $q(\varphi)$, $\gamma'(\varphi)$. Any one of these may be eliminated by use of the equality $\theta = \alpha+\gamma$. We shall be interested mainly in the normal accelerations and pitching velocities experienced by the airplane, and hence choose to eliminate α. Further, it will be found advantageous to separate out the steady-state values of the indicial functions. Thus, let

$$\left.\begin{aligned} C_{L_\alpha}(\varphi) &= C_{L_\alpha}(\infty) - F_1(\varphi) \\ C_{L_q}(\varphi) &= C_{L_q}(\infty) - F_2(\varphi) \\ C_{m_\alpha}(\varphi) &= C_{m_\alpha}(\infty) - F_3(\varphi) \\ C_{m_q}(\varphi) &= C_{m_q}(\infty) - F_4(\varphi) \end{aligned}\right\} \qquad (3)$$

The quantities $C_{L_\alpha}(\infty)$, $C_{L_q}(\infty)$, $C_{m_\alpha}(\infty)$, and $C_{m_q}(\infty)$ are the steady-state values of their respective indicial variations, and hence are also equal to the stability derivatives bearing the same subscripts. Substituting the expressions (3) in (2), carrying the derivatives through the integrals, and using the facts that $\alpha(0) = q(0) = 0$, $\alpha'(\varphi) = q(\varphi) - \gamma'(\varphi)$, we get for the equations of motion

$$\left.\begin{aligned} \eta\gamma'(\varphi) = {} & C_{L_\alpha}(\infty)\int_0^\varphi [q(\xi)-\gamma'(\xi)]d\xi + C_{L_q}(\infty)q(\varphi) - {} \\ & \int_0^\varphi F_1(\xi)[q(\varphi-\xi)-\gamma'(\varphi-\xi)]d\xi - \int_0^\varphi F_2(\xi)q'(\varphi-\xi)d\xi + 1\alpha_g C_{L_g}(\varphi) \\ \zeta q'(\varphi) = {} & C_{m_\alpha}(\infty)\int_0^\varphi [q(\xi)-\gamma'(\xi)]d\xi + C_{m_q}(\infty)q(\varphi) - {} \\ & \int_0^\varphi F_3(\xi)[q(\varphi-\xi)-\gamma'(\varphi-\xi)]d\xi - \int_0^\varphi F_4(\xi)q'(\varphi-\xi)d\xi + 1\alpha_g C_{m_g}(\varphi) \end{aligned}\right\} \qquad (4)$$

Transformed equations of motion.- Equations (4) can be solved for the unknown quantities $\gamma'(\varphi)$ and $q(\varphi)$ by the use of Laplace transforms. With the notation

$$L[F(\varphi)] = \int_0^\infty e^{-s\varphi}F(\varphi)\,d\varphi \tag{5}$$

let

$$\left.\begin{aligned}
L[\gamma'(\varphi)] &= s\Gamma(s)\\
L[q(\varphi)] &= Q(s)\\
L[C_{L_g}(\varphi)] &= c_{L_g}(s)\\
L[C_{m_g}(\varphi)] &= c_{m_g}(s)\\
L[F_i(\varphi)] &= f_i(s), \qquad i = 1,2,3,4
\end{aligned}\right\} \tag{6}$$

Carrying out the transformations (6) in equation (4) and solving for $s\Gamma(s)$ and $Q(s)$, we get

$$\left.\begin{aligned}
\frac{s\Gamma(s)}{l\alpha_g} &= \frac{s\left[c_{L_g}(s)D(s)-c_{m_g}(s)B(s)\right]}{A(s)D(s)-B(s)C(s)}\\
\frac{Q(s)}{l\alpha_g} &= \frac{s\left[c_{m_g}(s)A(s)-c_{L_g}(s)C(s)\right]}{A(s)D(s)-B(s)C(s)}
\end{aligned}\right\} \tag{7}$$

where

$$A(s) = s[\eta-f_1(s)] + C_{L_\alpha}(\infty)$$

$$B(s) = s^2f_2(s) - s[C_{L_q}(\infty)-f_1(s)] - C_{L_\alpha}(\infty)$$

$$C(s) = -sf_3(s) + C_{m_\alpha}(\infty)$$

$$D(s) = s^2[\zeta+f_4(s)] - s[C_{m_q}(\infty)-f_3(s)] - C_{m_\alpha}(\infty)$$

<u>Stability derivative analysis</u>.- Since we shall be interested in comparing the transient responses that result from solution of the exact equations (7) with those resulting from approximate formulations, it is instructive at this point to obtain the transformed equations which derive from the approximate but more familiar stability derivative analysis. In the latter case the equations of motion (in terms of chord lengths of travel) may be written

$$\left.\begin{aligned} \eta\gamma'(\varphi) &= C_{L_\alpha}(\infty)\alpha(\varphi) + C_{L_{\dot\alpha}}\alpha'(\varphi) + C_{L_q}(\infty)q(\varphi) + 1\alpha_g C_{L_g}(\varphi) \\ \zeta q'(\varphi) &= C_{m_\alpha}(\infty)\alpha(\varphi) + C_{m_{\dot\alpha}}\alpha'(\varphi) + C_{m_q}(\infty)q(\varphi) + 1\alpha_g C_{m_g}(\varphi) \end{aligned}\right\} \tag{8}$$

where $C_{L_{\dot\alpha}}$ and $C_{m_{\dot\alpha}}$ are the (constant) stability derivatives due to uniform vertical acceleration. Usually, in stability analyses, the terms $C_{L_q}(\infty)$ and $C_{L_{\dot\alpha}}$ are discarded; they will be retained here, however, in order to demonstrate fully the degree of correspondence that exists between equations (4) and (8). Further, in some gust analyses, the indicial gust functions $C_{L_g}(\varphi)$ and $C_{m_g}(\varphi)$ are replaced by constants, equal to their respective steady-state values, $C_{L_\alpha}(\infty)$ and $C_{m_\alpha}(\infty)$. The effect of this approximation will be considered subsequently.

Taking Laplace transforms in equation (8), eliminating α as before, and solving for $s\Gamma_1(s)$ and $Q_1(s)$, we get

$$\left.\begin{aligned} \frac{s\Gamma_1(s)}{1\alpha_g} &= \frac{s\left[c_{L_g}(s)D_1(s) - c_{m_g}(s)B_1(s)\right]}{A_1(s)D_1(s) - B_1(s)C_1(s)} \\ \frac{Q_1(s)}{1\alpha_g} &= \frac{s\left[c_{m_g}(s)A_1(s) - c_{L_g}(s)C_1(s)\right]}{A_1(s)D_1(s) - B_1(s)C_1(s)} \end{aligned}\right\} \tag{9}$$

where

$$\begin{aligned} A_1(s) &= s(\eta + C_{L_{\dot\alpha}}) + C_{L_\alpha}(\infty) \\ B_1(s) &= -s\left[C_{L_q}(\infty) + C_{L_{\dot\alpha}}\right] - C_{L_\alpha}(\infty) \\ C_1(s) &= C_{m_\alpha}(\infty) + sC_{m_{\dot\alpha}} \\ D_1(s) &= s^2\zeta - s\left[C_{m_q}(\infty) + C_{m_{\dot\alpha}}\right] - C_{m_\alpha}(\infty) \end{aligned}$$

and the subscript 1 is intended to distinguish the quantities from their exact counterparts in equation (7).

In order to make clear the approximations that are implicit in the use of equations (9) let us now derive equations (9) from the exact equations (7). To do this, expand the exponential $e^{-s\varphi}$ in a power series in s in each of the functions $f_i(s)$ in equations (7). Thus, for example, for the function $f_3(s)$ we have

$$f_3(s) = \int_0^\infty F_3(\varphi)d\varphi - s\int_0^\infty \varphi F_3(\varphi)d\varphi + s^2\int_0^\infty \frac{\varphi^2}{2!}F_3(\varphi)d\varphi - \dots \tag{10}$$

Now it is known (cf. ref. 20) that the first term in this expansion is equal to $-C_{m_{\dot{\alpha}}}$, the pitching-moment coefficient proportional to the motion $\alpha'(\varphi) = \text{constant}$. By a slight extension of the development of reference 20, one can go on to show that the remaining coefficients in equation (10) are likewise stability derivatives corresponding in order to the motions $\alpha''(\varphi) = \text{constant}$, $\alpha'''(\varphi) = \text{constant}$, etc. The same is true for the other indicial function transforms, and in fact the following identities can be shown to hold:

$$\left.\begin{aligned}
\frac{\partial C_L}{\partial \alpha^{(n)}} &= (-1)^n\int_0^\infty \frac{\varphi^{n-1}}{(n-1)!}F_1(\varphi)d\varphi\,; \qquad n\geq 1\\
\frac{\partial C_L}{\partial q^{(n)}} &= (-1)^n\int_0^\infty \frac{\varphi^{n-1}}{(n-1)!}F_2(\varphi)d\varphi\\
\frac{\partial C_m}{\partial \alpha^{(n)}} &= (-1)^n\int_0^\infty \frac{\varphi^{n-1}}{(n-1)!}F_3(\varphi)d\varphi\\
\frac{\partial C_m}{\partial q^{(n)}} &= (-1)^n\int_0^\infty \frac{\varphi^{n-1}}{(n-1)!}F_4(\varphi)d\varphi
\end{aligned}\right\} \tag{11}$$

When the forms (10) and (11) are substituted in equations (7), comparison of the result with equations (9) reveals that the latter equations are the result of retaining only the constant terms of the expansions of $f_1(s)$ and $f_3(s)$.

One is tempted to propose that the transient responses resulting from solution of equations (9) would agree more closely with those of equations (7) if some of the higher-order terms were to be included in equations (9). Such a proposal is incorrect, however, for the following reason: Retention of terms of order s and higher raises the degree of the denominators of equations (9) and hence the number of roots. The number of requirements for stability is therefore also raised, and it can be shown that these added requirements are spurious. In fact, the only stability requirements that are consistent with those of the exact equations (7) are those which exist for the forms (9). Therefore, we shall consider that the correct form of the stability derivative equations is that which stands in equations (9), and we shall make our subsequent comparisons on this basis.[1]

Solutions for $\gamma'(\varphi)$ and $q(\varphi)$.- The inversions of equations (7) give the complete histories $\gamma'(\varphi)$ and $q(\varphi)$ subsequent to the airplane's first penetration of the sharp-edge gust. Unfortunately, for most cases of interest, the equations for the indicial functions are so complex as to preclude attempts at direct inversions of equations (7). Several alternative techniques have been proposed. One such technique, which appears attractive for application to supersonic speeds, should be specifically warned against. Since the supersonic indicial functions reach constant values in a finite time, all approximations which take this behavior into account result in transcendental Laplace transforms of the type $f(s)e^{-s\sigma}$, where σ is a constant. One is then tempted to expand the exponential function in a series of polynomials in s. It has been demonstrated (cf. refs. 21 and 22) that expansions of this sort, in which higher-order terms are retained, lead to fallacious results. An expansion technique that is valid and applicable to both subsonic and supersonic speeds is to put equations (7) into the form $g(s)/1+k(s)$, expand in the series, $g(s)[1-k(s)+k^2(s) - \ldots]$ and invert term by term. This gives a series of convolution integrals which can be shown to be equivalent to the Liouville expansion used by Lomax in reference 7. The method is easy to apply but becomes excessively tedious if more than three or four terms of the series are required to insure adequate convergence. Rather than adopt this method, we shall make use of one recently presented by Huss and Donegan in reference 23. Here it is necessary to have the response to harmonic inputs of the system with the desired transient response. Having this information, one can, by comparatively simple and rapid numerical operations, extract the transient response to any desired accuracy. Therefore, rather than attempt to calculate the transient responses directly from equations (7), we turn instead to the calculation of the harmonic responses.

Airplane Motions in Response to Harmonic Gusts

It is well known (cf., for example, ref. 24) that the harmonic response and the transient responses to step or impulsive inputs are closely related. Having given the Laplace transforms of the step responses in equations (7) we may write down the harmonic responses directly from them by means of the relations

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[1]If desired, one can retain the terms $\partial C_L/\partial q'$, $\partial C_m/\partial q'$ in equations (9) since their retention does not raise the degree of the denominators. Their effect, however, is negligible.

$$\left.\begin{aligned}\frac{\gamma'}{\alpha_g}(i\lambda) &= \left[\frac{s^2\Gamma(s)}{l\alpha_g}\right]_{s=i\lambda}\\ \frac{q}{\alpha_g}(i\lambda) &= \left[\frac{sQ(s)}{l\alpha_g}\right]_{s=i\lambda}\end{aligned}\right\} \tag{12}$$

where λ is the (dimensionless) frequency of the harmonic gust input.

Exact and approximate solutions.- Equations (12) are valid under the restriction that the Laplace transforms in equations (7) $s\Gamma(s)/l\alpha_g$ and $Q(s)/l\alpha_g$ represent stable transfer functions. Assuming this is true, we may write the exact harmonic responses as

$$\left.\begin{aligned}\frac{\gamma'}{\alpha_g}(i\lambda) &= \frac{-\lambda^2\left[c_{L_g}(i\lambda)D(i\lambda) - c_{m_g}(i\lambda)B(i\lambda)\right]}{A(i\lambda)D(i\lambda) - B(i\lambda)C(i\lambda)}\\ \frac{q}{\alpha_g}(i\lambda) &= \frac{-\lambda^2\left[c_{m_g}(i\lambda)A(i\lambda) - c_{L_g}(i\lambda)C(i\lambda)\right]}{A(i\lambda)D(i\lambda) - B(i\lambda)C(i\lambda)}\end{aligned}\right\} \tag{13}$$

where

$$\begin{aligned}A(i\lambda) &= i\lambda[\eta - f_1(i\lambda)] + C_{L_\alpha}(\infty)\\ B(i\lambda) &= -\lambda^2 f_2(i\lambda) - i\lambda[C_{L_q}(\infty) - f_1(i\lambda)] - C_{L_\alpha}(\infty)\\ C(i\lambda) &= -i\lambda f_3(i\lambda) + C_{m_\alpha}(\infty)\\ D(i\lambda) &= -\lambda^2[\zeta + f_4(i\lambda)] - i\lambda[C_{m_q}(\infty) - f_3(i\lambda)] - C_{m_\alpha}(\infty)\end{aligned}$$

Likewise, of course, the harmonic responses as derived from the stability derivative analysis carry over from equations (9):

$$\left.\begin{aligned}\frac{\gamma_1'}{\alpha_g}(i\lambda) &= \frac{-\lambda^2\left[c_{L_g}(i\lambda)D_1(i\lambda) - c_{m_g}(i\lambda)B_1(i\lambda)\right]}{A_1(i\lambda)D_1(i\lambda) - B_1(i\lambda)C_1(i\lambda)}\\ \frac{q_1}{\alpha_g}(i\lambda) &= \frac{-\lambda^2\left[c_{m_g}(i\lambda)A_1(i\lambda) - c_{L_g}(i\lambda)C_1(i\lambda)\right]}{A_1(i\lambda)D_1(i\lambda) - B_1(i\lambda)C_1(i\lambda)}\end{aligned}\right\} \tag{14}$$

where

$$A_1(i\lambda) = i\lambda\eta' + C_{L_\alpha}(\infty) \; ; \qquad \eta' = \eta + C_{L_{\dot\alpha}}$$

$$B_1(i\lambda) = -i\lambda[C_{L_q}(\infty)+C_{L_{\dot\alpha}}] - C_{L_\alpha}(\infty)$$

$$C_1(i\lambda) = i\lambda C_{m_{\dot\alpha}} + C_{m_\alpha}(\infty)$$

$$D_1(i\lambda) = -\lambda^2\zeta - i\lambda[C_{m_q}(\infty)+C_{m_{\dot\alpha}}] - C_{m_\alpha}(\infty)$$

and, as before, the subscript 1 distinguishes the quantities from their exact counterparts in equations (13).

Numerical solutions.- With the use of equations (13), it is possible to compute exactly the harmonic responses in normal acceleration and pitching velocity of those wings whose indicial lift and moment responses to step changes in gust velocity, angle of attack, and pitching velocity have been calculated. For supersonic speeds, the necessary solutions are available for the two-dimensional wing (refs. 25 and 26), the rectangular wing (refs. 27 and 28), and the supersonic-edged triangular wing (refs. 29 and 30). For use in equations (13) these solutions must be transformed to functions of λ by means of equation (5) (with $s=i\lambda$). In order to facilitate the use of equations (13), the transformations have been carried out for the three classes of wings. The results are compiled in Appendix A.

Comparison of Exact and Approximate Step and Harmonic Gust Responses

Having solutions for the exact harmonic responses and a convenient method of extracting from them the transient responses (ref. 23), we are now in a position to study the effects of some of the approximations and assumptions commonly made in gust analyses. In order to examine these effects under conditions where they might be expected to be important, numerical calculations for the step and harmonic gust responses have been carried out for a rectangular wing of aspect ratio 3 flying at Mach number 1.2. The reason for this choice of Mach number was that it is principally at speeds near the sonic speed that the indicial functions show large variations. Hence, approximations to the indicial functions would be expected to be least valid there. The rectangular wing was chosen for study because of the necessity of having available a complete set of theoretical indicial functions applicable to Mach numbers near unity. It is recognized that for tailless airplanes the delta plan form would have been a more appropriate choice than the rectangular wing that was selected. The numerical results are therefore somewhat unrealistic; nevertheless, the conclusions to be drawn here regarding the validity of approximate solutions may reasonably be expected to apply as well to other more practical plan forms.

Harmonic responses.- Figures 1 and 2 show the wing's normal acceleration and pitching velocity responses to harmonic gust inputs as calculated from the exact transfer functions, equations (13), and from the approximate stability derivative formulation, equations (14). The constants which define the airplane's inertial properties are $\eta' = 100$, $\zeta = 100$; the center of gravity is located at the wing leading edge. It is apparent that except in the vicinity of the peaks of the curves, the approximate stability derivative results provide excellent representations of the true variations. It will be recalled that there were essentially two assumptions involved in the use of the stability derivative expressions; first, that the contributions of the functions $f_2(i\lambda)$, $f_4(i\lambda)$ could be neglected, and second, that the contributions $f_1(i\lambda)$, $f_3(i\lambda)$ could be replaced by the constants $-C_{L_{\dot{\alpha}}}$ and $-C_{m_{\dot{\alpha}}}$. It has been determined that such differences as do appear between the exact and approximate results are caused almost exclusively by the second of these assumptions; hence, neglect of the contributions of the indicial functions due to pitching velocity is found to be justified. As for the discrepancies caused by the second assumption, it should be noted that the frequency band over which they occur is very narrow (approximately $0.1 < \lambda < 0.2$). The differences in areas beneath the exact and approximate curves in this band are quite small, and hence one may anticipate that the discrepancies evident on figures 1 and 2 will cause only insignificant differences in the corresponding transient responses.

Step responses.- Figures 3 and 4 show the transient responses in normal acceleration and pitching velocity to a step change in gust velocity as calculated from the harmonic responses and the use of reference 23. It is clear that in the important features of the curves, namely, the maximum magnitudes and the points at which they occur, the differences between the exact and approximate results are not significant: The values of φ at which the maximum values of normal acceleration and pitching velocity occur are given by the approximate results almost without error; the error in maximum normal acceleration is less than 10 percent, whereas the error in maximum pitching velocity is hardly measurable.

Approximations to gust functions.- It may be instructive to point out that the conclusion to be drawn from the preceding section, that the use of stability derivatives in place of indicial functions is valid, is more general than is first apparent. The denominators of equations (7) form the "characteristic" part of the transfer functions - that is, the part that does not change with the type of excitation function. Having shown that stability derivatives may be used in the denominators without serious damage to the responses, we may expect that similar good results will be found for the responses to many other types of excitation. The effects of approximations to the excitations themselves, however, (in this case the gust functions) still remain to be studied.

It has been customary in some gust analyses to replace the gust functions $C_{L_g}(\varphi)$ and $C_{m_g}(\varphi)$ with step functions, equal in magnitude to their respective steady-state values $C_{L_\alpha}(\infty)$ and $C_{m_\alpha}(\infty)$. Thus, in the transfer functions, equations (9), the transformed gust functions $c_{L_g}(s)$ and $c_{m_g}(s)$ are replaced by the terms $C_{L_\alpha}(\infty)/s$ and $C_{m_\alpha}(\infty)/s$. The advantage is, of course, that now equations (9) are easy to invert. The results for the transient responses are

$$\left.\begin{aligned} \frac{\gamma'}{l\alpha_g}(\varphi) &= \zeta C_{L_\alpha}(\infty)R'(\varphi) + \lambda_1 R(\varphi) \\ \frac{q}{l\alpha_g}(\varphi) &= \lambda_2 R(\varphi) \end{aligned}\right\} \tag{15}$$

where

$$R(\varphi) = \frac{1}{\zeta\eta' k} e^{-\mu\varphi}\sin k\varphi$$

$$\eta' = \eta + C_{L_{\dot\alpha}}$$

$$\mu = \frac{1}{2\zeta\eta'}\left[\zeta C_{L_\alpha} - \eta\left(C_{m_q}+C_{m_{\dot\alpha}}\right) + C_{L_q}C_{m_{\dot\alpha}} - C_{m_q}C_{L_{\dot\alpha}}\right]$$

$$k = \left(\frac{-\eta C_{m_\alpha} + C_{L_q}C_{m_\alpha} - C_{m_q}C_{L_\alpha}}{\zeta\eta'} - \mu^2\right)^{1/2}$$

$$\lambda_1 = C_{m_\alpha}\left(C_{L_q}+C_{L_{\dot\alpha}}\right) - C_{L_\alpha}\left(C_{m_q}+C_{m_{\dot\alpha}}\right)$$

$$\lambda_2 = \eta' C_{m_\alpha} - C_{L_\alpha}C_{m_{\dot\alpha}}$$

These results are compared with the exact transient responses also in figures 3 and 4. It is clear that in this case the errors, at least in the normal acceleration response, may be significant; the maximum value of normal acceleration, for example, is overestimated by 25 percent. The errors in pitching velocity (fig. 4) are not as large; however the nature of the response for small values of φ is not correctly preserved, so that large errors will appear in the pitching acceleration response. As a saving feature, it is noted that the errors caused by the substitution of constants for the gust functions will diminish in severity with increasing supersonic Mach numbers, since the gust function variations do in fact rapidly approach steps for larger Mach numbers.

Neglect of pitching.- Since the loads caused by normal accelerations are in general more severe than those caused by pitching, many investigators have neglected the airplane's pitching degree of freedom altogether and assumed that the airplane is free to move only in the vertical direction. In order to investigate the validity of the foregoing assumption, numerical calculations have been carried out for the same operating conditions as used above, considering that the airplane is now restrained from pitching.

The exact harmonic response may be derived from equations (13) simply by letting the inertia parameter ζ approach infinity. The result is

$$\frac{\gamma'}{\alpha_g}(i\lambda) = \frac{-\lambda^2 c_{L_g}(i\lambda)}{i\lambda[\eta - f_1(i\lambda)] + C_{L_\alpha}(\infty)} \tag{16}$$

Likewise, the harmonic response as obtained from the stability derivative formulation (eqs. (14)) is

$$\frac{\gamma_1'}{\alpha_g}(i\lambda) = \frac{-\lambda^2 c_{L_g}(i\lambda)}{i\lambda\left(\eta + C_{L_{\dot{\alpha}}}\right) + C_{L_\alpha}(\infty)} \tag{17}$$

The corresponding transient responses may again be extracted from these results by use of reference 23. Numerical results for the harmonic and transient responses are shown in figures 5 and 6.

Note first in both figures 5 and 6 that, as in the previous cases, the differences between the exact results and those obtained from the stability derivative formulation are not significant. Next, compare the transient response for the single degree of freedom motion (fig. 6) with the exact response when pitching is included (fig. 3). It is noted that neglect of the pitching degree of freedom causes the maximum value of normal acceleration to be overestimated, but by only about 12 percent. Hence, use of either of the single degree of freedom equation (16) or (17) to predict maximum normal acceleration is probably justified. It may be necessary to point out, however, that even though the effect of the pitching degree of freedom on the maximum value of normal acceleration is small, the use of equation (16) or (17) rather than (13) or (14) will generally prevent one from obtaining the true nature of the remainder of the transient response curve. The reason for this can be most clearly seen by comparing the denominator or characteristic part of equation (17) with that of the equation in which the pitching degree of freedom is included, equation (14). The denominator of equation (17) is of first degree in λ, whereas that of equation (14) will be found to be of second degree. The transient response corresponding to equation (17) will therefore always be a subsidence whereas the transient response corresponding to equation (14) will most generally be oscillatory.

Concluding remarks.- The major conclusion to be drawn from the results of this study is that the use of the stability derivative formulation of the equations of motion does not significantly impair the accuracy of the solutions for transient and harmonic responses. It should be pointed out that the test case chosen for study represents conditions under which the differences between the exact and approximate formulations might be expected to be most apparent. In view of the excellent agreement between the two results, it should be anticipated that the stability derivative formulation will yield accurate results over a wide range of operating conditions and for many other types of plan forms.

SYNTHESIS

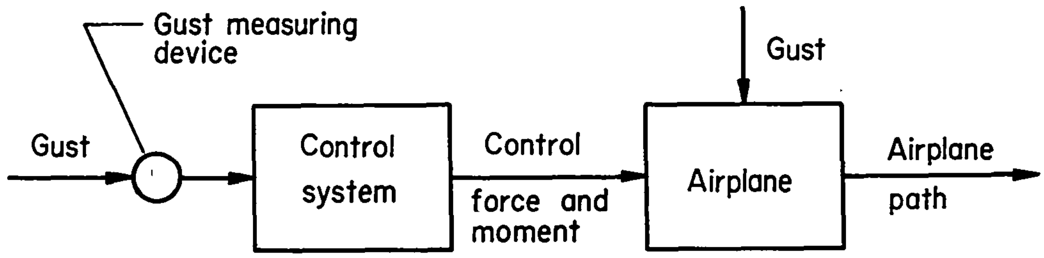
It is clear that since most gust disturbances are random in nature, this fact should be considered in a realistic analysis of the airplane's response to gusts. In recognition, a number of authors have studied the problem and have shown how the step and sinusoidal gust responses can be adapted to serve in the analysis of random motions as well (refs. 8, 9, and 10). The methods are based on concepts derived initially in the field of communications theory; their use is valid under the assumption that the disturbance mechanism can be described statistically as a stationary random process (ref. 24). As applied to the gust problem, the assumption appears to be a reasonable one in the light of evidence available from experimental studies of atmospheric turbulence (refs. 11, 12, and 13).

There is still another aspect of communications theory, however, valid within the same assumption and building on the same methods, which has not yet received wide attention in aerodynamic applications. This is the optimum filter theory of Wiener (ref. 18). The theory differs in emphasis from those mentioned earlier, in that the aim is to synthesize a system which minimizes responses to unwanted random disturbances; in contrast, the aim above is to analyze the responses of a given system. The connection between the aim of the Wiener theory and of gust alleviation is evident. In view of the gains possibly to be realized by devising a gust alleviation system which takes into explicit account the actual random nature of gust disturbances, consideration is given below to the adaptation of the Wiener theory to this end.

General Considerations

In the following general discussion of the gust alleviation problem several assumptions will be made regarding the availability of the quantities which are needed in order to define and solve the problem. It is believed that these assumptions are reasonable; however, in view of the exploratory nature of the study, questions concerning practical realizability of the quantities will not be considered.

Gust alleviation problem.- The problem to be considered can be illustrated by means of the block diagram in sketch (b). As indicated

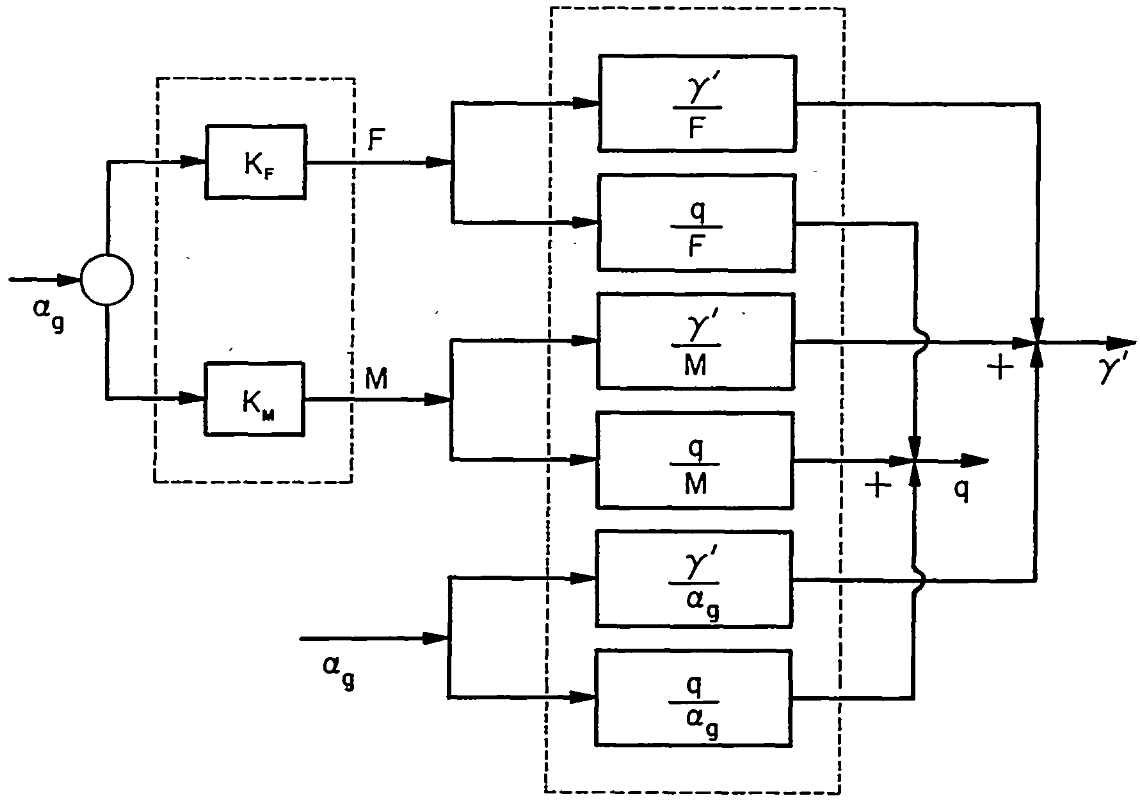


Sketch (b)

in the sketch, it is necessary that the airplane be equipped with a gust sensing device which sends to the control system a signal proportional to the gust velocity. It will be assumed that in response to commands from the control system there can be supplied a force and a moment which operate independently. It will also be assumed that the statistical nature of the gust disturbance is known, and that the responses of the airplane to step changes in the gust velocity and the control force and moment are known. Given this information, the problem is how to design the block labeled "control system" so that the force and moment it controls will counteract as nearly as possible the forces imposed by the gust so as to minimize the deviations of the airplane path from a desired path.

Here, it will be desired that the airplane fly a straight path, without pitching. It will be considered that the airplane motions most pertinent to the problem are the variations in normal acceleration and pitching velocity. With these specifications, a more explicit representation of the problem can be constructed as in sketch (c).

The block representing the control system in sketch (b) is now indicated by the two transfer functions K_F and K_M which send commands, respectively, to the control force and control moment. These commands are in response to signals from the gust sensing device. Similarly, the block representing the airplane in sketch (b) is indicated by the six transfer functions, relating inputs in control force, control moment, and gust velocity to the outputs in normal acceleration and pitching velocity. The net airplane motions γ' and q are of course the sums of the separate responses to the force, moment, and gust inputs; this is indicated in sketch (c) by the junctions marked with positive signs.



Sketch (c)

Equations of motion.- If we say that the blocks in sketch (c) are representative of the responses to step changes in the various inputs, then for arbitrary variations of the inputs, the system's equations of motion can be written by a straightforward application of the superposition integral. Thus, as functions of chord lengths of travel φ, the airplane motions $\gamma'(\varphi)$, $q(\varphi)$ are

$$\left.\begin{aligned}\gamma'(\varphi) &= \frac{d}{d\varphi}\int_0^\varphi \frac{\gamma'}{1F}(\varphi-\xi)F(\xi)d\xi + \frac{d}{d\varphi}\int_0^\varphi \frac{\gamma'}{1M}(\varphi-\xi)M(\xi)d\xi + \\ &\quad \frac{d}{d\varphi}\int_0^\varphi \frac{\gamma'}{1\alpha_g}(\varphi-\xi)\alpha_g(\xi)d\xi \\ q(\varphi) &= \frac{d}{d\varphi}\int_0^\varphi \frac{q}{1F}(\varphi-\xi)F(\xi)d\xi + \frac{d}{d\varphi}\int_0^\varphi \frac{q}{1M}(\varphi-\xi)M(\xi)d\xi + \\ &\quad \frac{d}{d\varphi}\int_0^\varphi \frac{q}{1\alpha_g}(\varphi-\xi)\alpha_g(\xi)d\xi\end{aligned}\right\} \quad (18)$$

Similarly, the control force and moment variations can be written

$$\left.\begin{aligned} F(\varphi) &= \frac{d}{d\varphi}\int_0^{\varphi} K_F(\varphi-\xi)\alpha_g(\xi)d\xi \\ M(\varphi) &= \frac{d}{d\varphi}\int_0^{\varphi} K_M(\varphi-\xi)\alpha_g(\xi)d\xi \end{aligned}\right\} \qquad (19)$$

The quantities $\frac{\gamma'}{1\alpha_g}(\varphi)$, $\frac{q}{1\alpha_g}(\varphi)$ in equations (18) are the responses in normal acceleration and pitching velocity to a step change in dimensionless gust velocity α_g, and hence are identical to the responses considered in detail earlier in this report. Likewise, the quantities $\frac{\gamma'}{1F}(\varphi)$, $\frac{\gamma'}{1M}(\varphi)$, $\frac{q}{1F}(\varphi)$, $\frac{q}{1M}(\varphi)$ are the responses to step changes in force and moment inputs; their derivation is entirely analogous to that of the gust responses. Finally, the quantities $K_F(\varphi)$, $K_M(\varphi)$ in equations (19) are the responses in control force and control moment, respectively, to a step change in the gust velocity. These quantities are the ones which are to be constructed such that the responses to random gusts are minimized.

Taking Laplace transforms in equations (18) and (19), we get

$$\left.\begin{aligned} s\Gamma(s) &= \left[\frac{s^2\Gamma(s)}{1F}\right] f(s) + \left[\frac{s^2\Gamma(s)}{1M}\right] m(s) + \left[\frac{s^2\Gamma(s)}{1\alpha_g}\right] A_g(s) \\ Q(s) &= \left[\frac{sQ(s)}{1F}\right] f(s) + \left[\frac{sQ(s)}{1M}\right] m(s) + \left[\frac{sQ(s)}{1\alpha_g}\right] A_g(s) \\ f(s) &= \left[sk_F(s)\right] A_g(s) \\ m(s) &= \left[sk_M(s)\right] A_g(s) \end{aligned}\right\} \qquad (20)$$

where

$$f(s) = L[F(\varphi)]$$

$$m(s) = L[M(\varphi)]$$

$$A_g(s) = L[\alpha_g(\varphi)]$$

$$k_i(s) = L[K_i(\varphi)], \qquad i = F,M$$

The bracketed quantities in equations (20) are the "system functions" of the various responses; they are equal to the transforms of the responses to impulsive inputs. Thus, for example, if $Q(s)/1F$ is the Laplace transform of the response in $q(\varphi)$ to a step change in F, then $sQ(s)/1F$ is the transform of the response in $q(\varphi)$ to an impulsive change in F. In order to simplify the notation, let the system functions be

$$\left.\begin{aligned} G_1(s) &= \frac{s^2\Gamma(s)}{1F} \qquad H_1(s) = \frac{sQ(s)}{1F} \\ G_2(s) &= \frac{s^2\Gamma(s)}{1M} \qquad H_2(s) = \frac{sQ(s)}{1M} \\ G_3(s) &= \frac{s^2\Gamma(s)}{1\alpha_g} \qquad H_3(s) = \frac{sQ(s)}{1\alpha_g} \\ X(s) &= sk_F(s) \\ Y(s) &= sk_M(s) \end{aligned}\right\} \tag{21}$$

Substituting equations (21) in (20) and solving for $s\Gamma(s)$ and $Q(s)$, we get

$$\left.\begin{aligned} s\Gamma(s) &= [G_1(s)X(s)+G_2(s)Y(s)+G_3(s)]A_g(s) \\ Q(s) &= [H_1(s)X(s)+H_2(s)Y(s)+H_3(s)]A_g(s) \end{aligned}\right\} \tag{22}$$

Perfect Alleviation

Before considering the application of Wiener theory to the gust alleviation problem it is instructive first to study the possibilities of perfect alleviation. By perfect alleviation we mean that the control system operates so as to reduce the airplane responses in normal acceleration and pitching velocity identically to zero for all, including random, gust inputs.

Reduction of motions to zero.- We note in equations (22) that, irrespective of the type of gust input, in principle it is possible to reduce identically to zero the airplane's response in both normal acceleration and pitching velocity. Thus, for any $A_g(s)$, $s\Gamma(s)$ and $Q(s)$ are zero if

$$\left.\begin{aligned} X(s) &= \frac{H_3(s)G_2(s)-H_2(s)G_3(s)}{G_1(s)H_2(s)-G_2(s)H_1(s)} \\ Y(s) &= \frac{H_1(s)G_3(s)-H_3(s)G_1(s)}{G_1(s)H_2(s)-G_2(s)H_1(s)} \end{aligned}\right\} \tag{23}$$

On reflection, this result is obvious; in effect it simply specifies that the control force and moment supplied in response to a step gust must exactly balance the force and moment imposed by the gust. Clearly, if the control system is capable of canceling the step gust input it will cancel all other gust inputs as well, since these may be viewed merely as successions of steps.

Alleviation with one control.- In view of the obvious complexity of a control system that might be built according to equations (23) we consider next the possibility of perfect alleviation when only one control is provided. Thus, we eliminate the independent control pitching moment $Y(s)$ in equations (22) and consider that only the control force (and its accompanying pitching moment) is available for counteracting the forces and moments imposed by the gust.

With $Y(s) = 0$, equations (22) become

$$\left.\begin{aligned} s\Gamma(s) &= [G_1(s)X(s)+G_3(s)]A_g(s) \\ Q(s) &= [H_1(s)X(s)+H_3(s)]A_g(s) \end{aligned}\right\} \tag{24}$$

Perfect alleviation is still a possibility in the event that

$$X(s) = -\frac{G_3(s)}{G_1(s)} = -\frac{H_3(s)}{H_1(s)} \tag{25}$$

In general, this will not be the case, as can be seen if it is recalled that in order to provide for perfect alleviation the force and moment imposed by a step gust must be canceled exactly. Now the force due to a step gust will in general build up monotonically from zero to a steady-state value whereas the pitching moment will change sign some time after the center of gravity of the airplane has passed through the gust front. Therefore, since for most types of aerodynamic controls the center of resolution of the force will remain essentially fixed, the single compensating force in general will not be capable of canceling both the gust force and gust moment simultaneously.

One such case may be said to exist, however; that is, when the gust force and moment build up so rapidly as to be essentially steps. Exact cancellation is then theoretically possible if the control force is placed at the airplane's aerodynamic center. This result (and also a result corresponding to eq. (23)) is in effect the solution given in reference 17. As indicated earlier in this report, the step approximation to the gust force and moment inputs should become increasingly valid as the flight Mach number increases sufficiently beyond unity, so that the analysis presented in reference 17 may apply even more appropriately to flight at very high speeds than it does for the speed range considered therein.

Application of Optimum Filter Theory

Although the solutions given by equations (23) and (25) are satisfactory from a theoretical standpoint, practically there are several objections to them. First, it is clear that the system's response must be both very rapid and precise in order to supply forces and moments matching those of the gust. Second, it may be argued that there is little point in attempting to build a system that in effect cancels gusts of all frequencies, since, even if no alleviation is provided, the airplane's inertia will prevent it from responding noticeably to gusts having frequencies very much larger than the airplane's natural frequency. Since we presume to possess some knowledge of the probable distribution of frequencies in the gust velocity, an alternate approach is to try to lighten the task of the control system by asking that it cancel only those frequency components of the gust velocity that actually do cause large normal accelerations and pitching velocities. The attractiveness of the Wiener theory of optimization is that it specifies just how to do this in order to satisfy a given criterion of excellence.

Responses to arbitrary control-force and gust velocity inputs.- In the subsequent study, we again assume that a single control force is available for counteracting the forces and moments imposed by gusts. Further, we shall assume that the step approximation to the gust functions is not valid so that the possibility of perfect alleviation is excluded.

Let us rewrite equations (24) as

$$\left.\begin{aligned} s\Gamma(s) &= L_1(s)A_g(s) \\ Q(s) &= L_2(s)A_g(s) \end{aligned}\right\} \tag{26}$$

where

$$L_1(s) = G_1(s)X(s)+G_3(s)$$

$$L_2(s) = H_1(s)X(s)+H_3(s)$$

It is clear from the form of equations (26) that the quantities $L_1(s)$ and $L_2(s)$ may be interpreted as over-all system functions; they relate the outputs in $\gamma'(\varphi)$ and $q(\varphi)$ to the gust input while the control is in operation.

A linear relationship exists between $L_1(s)$ and $L_2(s)$. By elimination of $X(s)$ it may be found to be

$$L_2(s) = \rho(s)L_1(s) + \sigma(s) \tag{27}$$

where

$$\rho(s) = \frac{H_1(s)}{G_1(s)}$$

$$\sigma(s) = \frac{G_1(s)H_3(s)-G_3(s)H_1(s)}{G_1(s)}$$

The quantities $\rho(s)$ and $\sigma(s)$ are of course known, involving as they do the known system functions $G_1(s)$, $H_1(s)$.

Now return to the physical domain. Let

$$\left.\begin{aligned} l_1(\varphi) &= L^{-1}[L_1(s)] \\ l_2(\varphi) &= L^{-1}[L_2(s)] \\ R(\varphi) &= L^{-1}[\rho(s)] \\ S(\varphi) &= L^{-1}[\sigma(s)] \end{aligned}\right\} \tag{28}$$

From equation (27)

$$l_2(\varphi) = L^{-1}[\rho(s)L_1(s)+\sigma(s)] \tag{29}$$

so that, letting

$$P(\varphi) = L^{-1}[\rho(s)L_1(s)] = \int_0^{\varphi} R(\xi)\, l_1(\varphi-\xi)d\xi \tag{30}$$

we have

$$l_2(\varphi) = P(\varphi) + S(\varphi) \tag{31}$$

Finally, from equations (26)

$$\left.\begin{aligned} \gamma'(\varphi) &= \int_0^{\infty} l_1(\xi)\alpha_g(\varphi-\xi)d\xi \\ q(\varphi) &= \int_0^{\infty} l_2(\xi)\alpha_g(\varphi-\xi)d\xi \end{aligned}\right\} \tag{32}$$

It will be noted in equations (32) that we have used a more general form of the superposition integral than heretofore. The reason for this is that we wish to invoke the assumption usually made in random processes, namely, that the process has started infinitely distant in the past. Consequently, for an arbitrary origin of φ, the existence of inputs for negative arguments must be admitted, and this is accomplished in equations (32) by letting the upper limits of the integrals approach infinity.

The error.- Equations (32) represent the variations in normal acceleration and pitching velocity caused by arbitrary inputs of gust velocity and the unknown compensating force. It is desired to minimize these variations according to some criterion when the gust inputs are random. We shall take as our error criterion a linear combination of the mean-square values of the variations $\gamma'(\varphi)$ and $q(\varphi)$ over the infinite interval. We choose to minimize a combination of the motions in order to reduce the possibility of arriving only at solutions which minimize one motion at the risk of intolerably increasing the other. Thus, we seek to minimize the mean-square error

$$\overline{\epsilon^2} = \overline{(q^2)} + \frac{1}{a}\overline{(\gamma'^2)}; \qquad a > 0 \tag{33}$$

where a is a weighting factor, open to choice depending on which it is thought to be more important to reduce, γ' or q.

The choice of a mean-square error criterion is of course somewhat arbitrary. Criteria other than the mean square are possible and may lead to greater reductions in the variations $\gamma'(\varphi)$ and $q(\varphi)$. However, its use in connection with the gust alleviation problem is reasonable, in view of its property of weighting the minimization in favor of reducing the undesirable large errors while permitting many small ones. Also, its form is mathematically convenient, since it leads to an integral equation that can be solved by known techniques.

Integral representation of the error.- Substitution of equations (32) in (33) gives for the mean-square error

$$\overline{\epsilon^2} = \overline{\left[\int_0^\infty l_2(\xi)\alpha_g(\varphi-\xi)d\xi\right]^2} + \frac{1}{a}\overline{\left[\int_0^\infty l_1(\xi)\alpha_g(\varphi-\xi)d\xi\right]^2} \tag{34}$$

Expanding the integrals, we have

$$\overline{\epsilon^2} = \overline{\left[\int_0^\infty l_2(\xi_1)\alpha_g(\varphi-\xi_1)d\xi_1\int_0^\infty l_2(\xi_2)\alpha_g(\varphi-\xi_2)d\xi_2\right]} + \frac{1}{a}\overline{\left[\int_0^\infty l_1(\xi_1)\alpha_g(\varphi-\xi_1)d\xi_1\int_0^\infty l_1(\xi_2)\alpha_g(\varphi-\xi_2)d\xi_2\right]} \tag{35}$$

Now by definition

$$\overline{f(\varphi)} = \lim_{\Phi\to\infty}\frac{1}{2\Phi}\int_{-\Phi}^{\Phi} f(\varphi)d\varphi \tag{36}$$

Then in equation (35)

$$\overline{\epsilon^2} = \lim_{\Phi\to\infty}\frac{1}{2\Phi}\int_{-\Phi}^{\Phi}\left[\int_0^\infty l_2(\xi_1)d\xi_1\int_0^\infty l_2(\xi_2)\alpha_g(\varphi-\xi_1)\alpha_g(\varphi-\xi_2)d\xi_2 + \frac{1}{a}\int_0^\infty l_1(\xi_1)d\xi_1\int_0^\infty l_1(\xi_2)\alpha_g(\varphi-\xi_1)\alpha_g(\varphi-\xi_2)d\xi_2\right]d\varphi \tag{37}$$

The limiting process acts only on $\alpha_g(\varphi-\xi_1)$ since all other terms are functions only of ξ_1, ξ_2. We define

$$\lim_{\Phi\to\infty} \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} \alpha_g(\varphi-\xi_1)\alpha_g(\varphi-\xi_2)d\varphi = W(\xi_1-\xi_2) \tag{38}$$

This is the autocorrelation function of the random distribution of gust disturbances. We assume that the function exists, and moreover, that it is known. Carrying the limit through the integrals in equation (37) and using (38) we get for the mean-square error

$$\begin{aligned}\overline{\epsilon^2} = &\int_0^\infty l_2(\xi_1)d\xi_1 \int_0^\infty l_2(\xi_2)W(\xi_1-\xi_2)d\xi_2 + \\ &\frac{1}{a}\int_0^\infty l_1(\xi_1)d\xi_1 \int_0^\infty l_1(\xi_2)W(\xi_1-\xi_2)d\xi_2\end{aligned} \tag{39}$$

Note that the mean-square error does not depend on the gust velocities themselves, but rather only on their correlation function.

Equation (39) can be cast in terms of one unknown function $P(\varphi)$ by means of equations (30) and (31). Thus, in equation (30), let

$$R_0(\varphi) = L^{-1}\left[\frac{1}{\rho(s)}\right] \tag{40}$$

Then

$$l_1(\varphi) = \int_0^\varphi P(\xi)R_0(\varphi-\xi)d\xi \tag{41}$$

The upper limit φ is justified here since $P(\varphi)$, $R_0(\varphi)$, and $l_1(\varphi)$ are impulse responses and hence must be zero for negative arguments. In addition, let

$$U(\xi_1) = \int_0^\infty S(\xi_2)W(\xi_1-\xi_2)d\xi_2 \tag{42}$$

Substituting equations (31), (41), and (42) in (39), we get

$$\overline{\epsilon^2} = \int_0^\infty S(\xi_1)U(\xi_1)d\xi_1 + 2\int_0^\infty P(\xi_1)U(\xi_1)d\xi_1 +$$

$$\int_0^\infty P(\xi_1)d\xi_1 \int_0^\infty P(\xi_2)W(\xi_1-\xi_2)d\xi_2 +$$

$$\frac{1}{a}\int_0^\infty d\xi_1 \int_0^{\xi_1} P(\sigma_1)R_o(\xi_1-\sigma_1)d\sigma_1 \int_0^\infty W(\xi_1-\xi_2)d\xi_2 \int_0^{\xi_2} P(\sigma_2)R_o(\xi_2-\sigma_2)d\sigma_2 \tag{43}$$

Now in the quadruple integral, reversing the order of integration results in

$$\int_0^\infty P(\sigma_1)d\sigma_1 \int_0^\infty P(\sigma_2)d\sigma_2 \int_{\sigma_1}^\infty R_o(\xi_1-\sigma_1)d\xi_1 \int_{\sigma_2}^\infty R_o(\xi_2-\sigma_2)W(\xi_1-\xi_2)d\xi_2$$

Interchanging dummy variables ($\xi_1 \longleftrightarrow \sigma_1$, $\xi_2 \longleftrightarrow \sigma_2$), we have

$$\int_0^\infty P(\xi_1)d\xi_1 \int_0^\infty P(\xi_2)d\xi_2 \int_{\xi_1}^\infty R_o(\sigma_1-\xi_1)d\sigma_1 \int_{\xi_2}^\infty R_o(\sigma_2-\xi_2)W(\sigma_1-\sigma_2)d\sigma_2$$

$$= \int_0^\infty P(\xi_1)d\xi_1 \int_0^\infty P(\xi_2)d\xi_2 \int_0^\infty R_o(\sigma_1)d\sigma_1 \int_0^\infty R_o(\sigma_2)W(\xi_1-\xi_2+\sigma_1-\sigma_2)d\sigma_2 \tag{44}$$

It is clear from the second of the above forms that the result of the double integration involving $R_o(\varphi)$ is a function of $(\xi_1-\xi_2)$ alone. Finally, using the second of the forms (44), we combine the last two terms in equation (43); this gives

$$\int_0^\infty P(\xi_1)d\xi_1 \int_0^\infty P(\xi_2)d\xi_2 \left[W(\xi_1-\xi_2) + \frac{1}{a}\int_0^\infty R_o(\sigma_1)d\sigma_1 \int_0^\infty R_o(\sigma_2)W(\xi_1-\xi_2+\sigma_1-\sigma_2)d\sigma_2\right] \tag{45}$$

The bracketed quantity in (45) can be shown to possess all the properties of an autocorrelation function (ref. 24). It is, in fact, necessary that this be true in order for our subsequent development to hold. Let this function be called $\Psi(\xi_1-\xi_2)$. Then, returning to equation (43), we find that the mean-square error becomes

$$\overline{\epsilon^2} = \int_0^\infty S(\xi_1)U(\xi_1)d\xi_1 + 2\int_0^\infty P(\xi_1)U(\xi_1)d\xi_1 + \int_0^\infty P(\xi_1)d\xi_1 \int_0^\infty P(\xi_2)\Psi(\xi_1-\xi_2)d\xi_2 \tag{46}$$

Integral equation.- Since S, U, and Ψ are known functions (cf. eqs. (28), (42), and (45)) while P is unknown (eq. (30)), equation (46) is now of precisely the form considered by Wiener in reference 18. As shown therein by an application of the calculus of variations, the mean-square error is a minimum if (and only if) $P(\varphi)$ is constructed such that the following relation is satisfied:

$$U(\varphi) + \int_0^\infty P(\xi_1)\Psi(\varphi-\xi_1)d\xi_1 = 0 \; ; \quad \varphi \geq 0 \tag{47}$$

Equation (47) is known as a Wiener-Hopf integral equation of the first kind for the unknown $P(\varphi)$. A general analytical method for solving it is described in reference 18 (cf., in particular, Appendix C of ref. 18). Also, several numerical methods now exist (cf., for example, refs. 31 and 32) which can be used to solve equation (47) approximately should an analytical approach prove impracticable. Hence, we can assume that it is possible to extract $P(\varphi)$ from equation (47). Having $P(\varphi)$, we may compute $l_2(\varphi)$ from equation (31). The Laplace transform of the control system necessary to achieve a minimum mean-square error is then obtainable from the relation (see eq. (26))

$$X(s) = \frac{L_2(s)-H_3(s)}{H_1(s)} \tag{48}$$

This is the sought-for quantity - the transfer function of the response in control force to gust inputs.

The minimized mean-square error, which results when the control-force system is constructed according to equation (48) may be found conveniently by substituting equation (47) into (46) and using equation (31). The result is

$$\overline{\epsilon^2}_{min} = \int_0^\infty U(\xi_1)\, l_2(\xi_1)d\xi_1 \tag{49}$$

Finally, in order to compare the minimized mean-square error with the error that results when no alleviation is provided (i.e., $X(s)=0$), it will be necessary to have the latter result. Setting $X(s)$ equal to zero in equations (26) and using equation (39), we have for the unmodified mean-square error

$$\overline{\epsilon^2} = \int_0^\infty h_3(\xi_1)d\xi_1 \int_0^\infty h_3(\xi_2)W(\xi_1-\xi_2)d\xi_2 + \frac{1}{a}\int_0^\infty g_3(\xi_1)d\xi_1 \int_0^\infty g_3(\xi_2)W(\xi_1-\xi_2)d\xi_2 \tag{50}$$

where

$$h_3(\varphi) = L^{-1}[H_3(s)]$$

$$g_3(\varphi) = L^{-1}[G_3(s)]$$

Application of Results

It remains to investigate some of the characteristics of a control system that might be built according to the specifications of the theory just presented. To be acceptable, one should expect of such a system at least the following: (1) adequate stability; (2) practical feasibility; (3) a significant reduction in the mean-square error from the error that results when no alleviation is provided. Unfortunately, these requirements are not necessarily compatible; in order to check them numerical analyses must be made, using specific airplanes and operating conditions. We have chosen as an example case a tailless triangular-wing airplane flying at a high subsonic Mach number.

System functions.- Since we have determined in the first part of this paper that the stability derivative formulation is a valid approximation to the exact equations of motion, we shall use it here to define the airplane system functions. Further, in the system functions $G_1(s)$ and $H_1(s)$, we shall assume that the location of the control force remains essentially fixed. Thus, for a step control force $1F$, the control moment is $-1Fx_o$, where x_o is the (dimensionless) distance of the point of application of the force from the airplane center of gravity. With these assumptions, the system functions may be derived from equations (9) and (21). (The functions $G_1(s)$ and $H_1(s)$ are obtained simply by replacing $c_{L_g}(s)$ and $c_{m_g}(s)$ in equations (9) respectively by $1/s$ and $-(x_o/s)$.) The results are

$$\left.\begin{aligned}
G_1(s) &= \frac{s^2\Gamma(s)}{lF} = \frac{\zeta s^2 - s\left[C_{m_q}+C_{m_{\dot\alpha}}+x_o\left(C_{L_q}+C_{L_{\dot\alpha}}\right)\right] - \left(x_oC_{L_\alpha}+C_{m_\alpha}\right)}{\zeta\eta'[(s+\mu)^2+k^2]} \\
G_3(s) &= \frac{s^2\Gamma(s)}{l\alpha_g} = \frac{s\left\{c_{L_g}(s)\left[\zeta s^2 - s\left(C_{m_q}+C_{m_{\dot\alpha}}\right) - C_{m_\alpha}\right] + c_{m_g}(s)\left[s\left(C_{L_q}+C_{L_{\dot\alpha}}\right)+C_{L_\alpha}\right]\right\}}{\zeta\eta'[(s+\mu)^2+k^2]} \\
H_1(s) &= \frac{sQ(s)}{lF} = \frac{-s\left(\eta' x_o+C_{m_{\dot\alpha}}\right) - \left(C_{m_\alpha}+x_oC_{L_\alpha}\right)}{\zeta\eta'[(s+\mu)^2+k^2]} \\
H_3(s) &= \frac{sQ(s)}{l\alpha_g} = \frac{s\left[c_{m_g}(s)\left(\eta' s+C_{L_\alpha}\right) - c_{L_g}(s)\left(sC_{m_{\dot\alpha}}+C_{m_\alpha}\right)\right]}{\zeta\eta'[(s+\mu)^2+k^2]}
\end{aligned}\right\} \quad (51)$$

where

$$\eta' = \eta + C_{L_{\dot\alpha}}$$

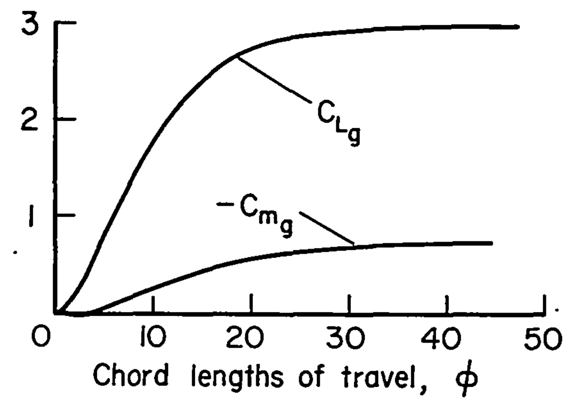
$$\mu = \frac{1}{2\zeta\eta'}\left[\zeta C_{L_\alpha} - \eta\left(C_{m_q}+C_{m_{\dot\alpha}}\right) + C_{L_q}C_{m_{\dot\alpha}} - C_{m_q}C_{L_{\dot\alpha}}\right]$$

$$k = \left(\frac{-\eta C_{m_\alpha}+C_{L_q}C_{m_\alpha}-C_{m_q}C_{L_\alpha}}{\zeta\eta'} - \mu^2\right)^{1/2}$$

It remains to define the gust functions $C_{L_g}(\varphi)$ and $C_{m_g}(\varphi)$ whose Laplace transforms appear in $G_3(s)$ and $H_3(s)$. In the absence of theoretical results for these functions applicable to subsonic speeds, we shall assume they may be approximated by the expressions

$$\left.\begin{aligned}
C_{L_g}(\varphi) &= C_{L_\alpha}(\infty)\left[1-e^{-p\varphi}(1+p\varphi)\right] \\
C_{m_g}(\varphi) &= C_{m_\alpha}(\infty)\left[1-e^{-p\varphi}(1+p\varphi+r\varphi^2)\right]
\end{aligned}\right\} \quad (52)$$

where p and r are parameters. It is emphasized that these are not the gust functions of any known wing. They are considered to be physically reasonable, however, and, by virtue of their zero slope at $\varphi = 0$, are believed to be specifically representative of pointed-nose wings. They are strictly applicable only to flight at subsonic Mach numbers, in view of their asymptotic approach to steady-state values. The particular Mach number to which the functions correspond may be varied by changing $C_{L_\alpha}(\infty)$, $C_{m_\alpha}(\infty)$, p, and r. As the flight Mach number is increased toward unity p should be reduced in value. The value of r should be chosen so that $C_{m_g}(\varphi)$ has its greatest positive value near the value of φ for which the airplane center of gravity passes through the gust front. The particular functions which were used in numerical computations are plotted in sketch (d).



Sketch (d)

Finally, the Laplace transforms of equations (52), which are to be inserted in the expressions for $G_3(s)$ and $H_3(s)$, are

$$\left.\begin{aligned} c_{L_g}(s) &= C_{L_\alpha}(\infty)\left[\frac{p^2}{s(s+p)^2}\right] \\ c_{m_g}(s) &= C_{m_\alpha}(\infty)\left[\frac{p^2(s+p)-2rs}{s(s+p)^3}\right] \end{aligned}\right\} \tag{53}$$

Stability and the location of control force.- Having defined the system functions, we next consider the conditions which must be imposed on them to insure the stability of the airplane and control system. First, it is a necessity that the system functions themselves be stable. In order to insure their stability, it is sufficient to stipulate that their denominators contain no zeroes having positive real parts. This will be true in all four functions provided the damping factor μ and the stiffness factor k^2 are positive. These requirements are of course the usual ones that arise in nearly all dynamic stability analyses.

In addition, we must insure the stability of both $\rho(s)$ and its reciprocal, since both enter into the analysis for the optimum control force (see eqs. (30) and (40)). Hence, it is a requirement that both numerator and denominator of $\rho(s)$ be free of zeroes having positive real parts. Now $\rho(s)$ may be written in the form (cf. eqs. (27) and (51))

$$\rho(s) = \frac{H_1(s)}{G_1(s)} = \frac{1}{K}\,\frac{(s+c_1)}{(s^2+2\mu_1 s+k_1{}^2)} \quad .$$

where

$$K = -\frac{\zeta}{\eta' x_o+C_{m_{\dot{\alpha}}}}$$

$$c_1 = \frac{x_o C_{L_\alpha}+C_{m_\alpha}}{\eta' x_o+C_{m_{\dot{\alpha}}}}$$

$$2\mu_1 = -\frac{1}{\zeta}\left[\left(C_{m_q}+C_{m_{\dot{\alpha}}}\right) + x_o\left(C_{L_q}+C_{L_{\dot{\alpha}}}\right)\right]$$

$$k_1{}^2 = -\frac{1}{\zeta}\left(C_{m_\alpha}+x_o C_{L_\alpha}\right)$$

Then $\rho(s)$ and $1/\rho(s)$ are stable if c_1, μ_1, and $k_1{}^2$ are all positive. We note in equation (54) that c_1, μ_1, and $k_1{}^2$ are functions of x_o, the control location, and hence the requirements of stability restrict to a great extent the possible locations of the control force. Thus, from equation (54)

$$\left.\begin{aligned} k_1{}^2 > 0 \qquad &\text{if} \quad x_o < -\frac{C_{m_\alpha}}{C_{L_\alpha}} \\ \mu_1 > 0 \qquad &\text{if} \quad x_o < -\frac{C_{m_q}+C_{m_{\dot{\alpha}}}}{C_{L_q}+C_{L_{\dot{\alpha}}}} \end{aligned}\right\} \tag{55}$$

With $k_1{}^2 > 0$, we have

$$c_1 > 0 \qquad \text{if} \quad x_o < -\frac{C_{m_{\dot{\alpha}}}}{\eta'} \tag{56}$$

The condition $k_1{}^2 > 0$ excludes as possible locations of the control force all points aft of the aerodynamic center x_a; whereas, with $k_1{}^2 > 0$, in order to have $c_1 > 0$ we must exclude essentially all points aft of the center of gravity. With $k_1{}^2 = c_1 = 0$, the aerodynamic center is a possible location in the event that $x_a < -[(C_{m_q}+C_{m_{\dot{\alpha}}})/(C_{L_q}+C_{L_{\dot{\alpha}}})]$. This cannot be considered a satisfactory position, however, since only a slight shift of x_o in either direction from x_a will cause c_1 to be negative in one case or $k_1{}^2$ in the other. Hence, we conclude that in order to satisfy the requirements for stability, the control force must be located forward of the center of gravity and sufficiently so to insure that $x_o < -(C_{m_{\dot{\alpha}}}/\eta')$.

Gust correlation function.- One further piece of information needs to be supplied before the analysis for the optimum control system can be carried out; that is, the specification of the gust correlation function $W(\varphi)$. We shall use for this quantity an expression that has been adopted by a number of authors in recent applications to gust analyses (refs. 10 and 33 through 36), namely

$$W(\varphi) = \overline{\alpha_g^2}\left(1 - \frac{c}{2L}|\varphi|\right) e^{-\frac{c}{L}|\varphi|} \tag{57}$$

Here, $\overline{\alpha_g^2}$ is the average intensity of the gust vertical velocities, $\overline{w_g^2}$, made dimensionless with respect to the airplane forward speed V. The quantity L is a measure of the so-called "scale of turbulence" (ref. 34). Comparisons of equation (57) with the results of existing experimental studies have indicated that the value of L may vary from the order of several hundred feet to over a thousand feet. The reader is referred to references 10, 34, and 36 for a more detailed explanation of the origin of equation (57) and the ranges of values to be expected of the quantities $\overline{w_g^2}$ and L.

Numerical results.- Equations (51), (52), and (57) complete the specification of quantities necessary for computation of the optimum control system from equation (48) and the minimized mean-square error from equation (49). Detailed results of the analysis are presented for the general case in Appendix B. In this section curves will be presented which illustrate the nature of the results as they apply to a single set of operating conditions. The numerical values which define the airplane's inertial and geometric properties, its transfer functions, and the gust correlation function are listed at the end of Appendix B. The set of constants defining the airplane are believed to be representative of those corresponding to a fighter-type triangular-wing airplane flying at a high subsonic speed. The constant L used in the gust correlation function corresponds to a scale of turbulence of 300 feet. The constant $\overline{\alpha_g^2}$, defining the average intensity of turbulence, was left unspecified since it appears in the results for mean-square error only as a multiplying factor. Hence, the results can be made applicable to a range of weather conditions by appropriate selections of this parameter.

Mean-square error: Figure 7 is a comparison of the magnitudes of the minimized and unmodified mean-square errors (eqs. (49) and (50)) for a range of values of weighting factor a and several values of force position. Large values of a correspond to weighting the minimization procedure in favor of reducing the mean-square pitching velocity, possibly at the risk of increasing the mean-square normal acceleration beyond its unmodified value. For small values of a the reverse is true. It is clear from inspection of the figure that the reductions in total error may be significant; the results are to be interpreted, however, as indicating only that a significant reduction in one or the other of the individual components of the mean-square error is possible. It remains to be seen whether such can be realized without the penalty of increasing one component beyond its unmodified value. This point is clarified in figure 8, which shows the individual components of the mean-square error given as ratios of their corresponding unalleviated values. (The note at the top of figure 8 permits one to attach absolute values to the individual components of the mean-square error, should this be desired.) It is clear that either error can be made arbitrarily small, but eventually at the expense of increasing the other beyond its unmodified value. Of greater importance is that a range of a does exist for which significant reductions in both errors are possible at the same time. Thus, for example, for the control force located at the wing nose ($x_o = -0.25$), both errors may be reduced to about half their unalleviated values when a is near 0.1.

In addition, note that the control force becomes even more effective in reducing both errors as its point of application is moved forward of the nose. This would suggest that a canard control surface might be an efficient means of generating the required force. It should be pointed out, however, that the analysis in its present form does not take account of the effects on the lift of the main wing of downwash from the canard, so that the analysis is applicable to the canard only if these effects can be considered negligible. As previously mentioned, the entire question of how to generate the required control force is considered to be beyond the scope of this paper, and the present results are of value mainly as an indication that more extensive analyses, in which this question must of course be considered, may be justified.

Control force: We examine next the properties of the control force that achieves reductions of both errors, using as an example the case $x_o = -0.25$, that is, where the force is located at the wing nose.

Figure 9 shows the transient response of the control force to a step gust. This force caused the mean-square values of normal acceleration and pitching velocity to be reduced to about half their unalleviated values. Also shown as dashed curves in figure 9 are the inputs in aerodynamic force and moment due to a step gust. Note that the control force need be only about a third the magnitude of the gust force and need be applied much less rapidly. This should be contrasted with the problem of devising a system to provide perfect alleviation, for which the control force and moment must be made to match those of the gust at every instant.

Further insight into the nature of the control force may be gained from inspection of the amplitudes of the control-force response to harmonic gusts. These are shown in figure 10. Also shown in figure 10 is a plot of the gust spectral density, normalized so that its value is unity at $\lambda = 0$. (The spectral density is defined as the Fourier transform of the gust correlation function, eq. (57); it is a measure of the "energy" contained in each increment of frequency in the gust spectrum (cf. ref. 8).) Note that, as might be anticipated, the amplitude of control force falls off with increasing frequency as the gust spectral density does. Again, this should be contrasted to the dashed curve which is the amplitude of control force required for perfect alleviation. In the latter case the force is independent of the gust characteristics and clearly is of significant magnitude over a wider frequency range.

Airplane responses: Figures 11 and 12 show the amplitudes of the responses in normal acceleration and pitching velocity to harmonic gusts with and without operation of the control. As in figures 9 and 10, the control force is located at the wing apex. On comparison of the results for $a = 0.1$ with the unmodified responses, it is seen in both figures 11 and 12 that the amplitudes of the optimized responses have been significantly reduced in the vicinity of the peaks of the curves and that the peaks themselves have been shifted to the right where the gust spectral density (fig. 10) has begun to fall off. The effect of both modifications is of course to reduce the mean-square errors. Also shown in figure 11 is the response in normal acceleration required to give zero pitching velocity response ($a \rightarrow \infty$) and in figure 12, the response in pitching velocity which gives zero normal acceleration ($a \rightarrow 0$). The reasons for the increase in mean-square error of one response, which is a consequence of attempting to minimize only the other response, are readily apparent from a comparison of the two curves with their respective unmodified responses.

As a final point, it will be noted in figures 11 and 12 that the initial values of the optimum harmonic responses are not zero. This means that the steady-state values of the corresponding transient responses to a step gust are not zero. Hence, in response to a step gust, the control force will cause the airplane eventually to climb or dive. Unfortunately, the worsening of the transient response usually must be accepted as a consequence of attempting to minimize the response to continuous random disturbances; one must count on the probability that isolated step gusts will not be encountered.

Off-design performance: The results presented in figures 7 through 12 apply only to a single atmospheric condition, namely, that corresponding to a scale of turbulence of 300 feet. It is of interest to ask how the system designed to operate optimally at one atmospheric condition will perform at others. To investigate this, some computations have been carried out for the control force which is optimum for a scale of turbulence of 1000 feet. The results are shown in figure 13 in comparison with the optimum control force for $L = 300$ feet (from fig. 9). It is apparent that, since the results are quite similar, the design is not strongly dependent on the magnitude of L. Hence, one may expect that a system designed to be optimum for one atmospheric condition will perform well for off-design conditions not too distant from the design condition.

The atmospheric conditions are also dependent on the quantity $\overline{\alpha_g^2}$. However, this quantity occurs in the expressions for both the minimized and unmodified errors as a scale factor. Hence, changing $\overline{\alpha_g^2}$ will have the same effect on both of these errors and the errors of the optimized system will be reduced by the same percentage for all values of $\overline{\alpha_g^2}$.

Concluding Remarks

The preceding application of the Wiener optimum filter theory to the problem of reducing the airplane's response to random atmospheric turbulence has demonstrated that such an approach may lead to useful results. It was found that significant reductions of both the normal acceleration and pitching velocity responses may be achieved, and that construction of a control system which realizes these improvements may be considerably simpler than construction of one designed to provide perfect alleviation. Nevertheless, the method and results presented here are by no means meant to be final or definitive. For one thing, the problem for tailed aircraft has not been touched; the revisions and amendments of the analysis which are necessary to account for the tail and the effects of downwash may invalidate some of the results obtained here. Further, even for tailless aircraft, in some cases the assumption that the control pitching moment is proportional to the control force may not be warranted. This would probably be the case, for example, in the event that the force is to be provided by a canard control surface whose interference effects are not negligible. Moreover, other formulations of the problem are possible. For example, rather than leave the control force unspecified as was done here, it may turn out to be practically more feasible to begin with a control system of known type having, say, open parameters and use the Wiener theory to give optimum values to these parameters. The main function of the present work may be to offer evidence that efforts in these directions will likewise yield results that indicate the possibility of achieving significant reductions of the airplane's response to random turbulence.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., June 28, 1957

APPENDIX A

TRANSFORMATIONS TO HARMONIC RESPONSES OF THE SUPERSONIC INDICIAL FUNCTIONS OF TWO-DIMENSIONAL, RECTANGULAR, AND WIDE TRIANGULAR WINGS

The following is a compilation of the transformations to harmonic responses of the supersonic indicial lift and pitching-moment responses to step changes in (1) gust velocity, (2) angle of attack, and (3) pitching velocity. The transforms are presented for the two-dimensional wing, the rectangular wing, and the wide triangular wing. Sources from which the indicial functions were obtained are listed as references 25 to 30.

NOTATION AND TRANSFER OF AXES

All coefficients to be presented are given as transforms of deficiency functions; these are defined as follows:

$$\left.\begin{aligned}
g_1(i\lambda) &= \int_0^\infty \left[C_{L_g}(\infty)-C_{L_g}(\varphi)\right] e^{-i\lambda\varphi}d\varphi \\
g_3(i\lambda) &= \int_0^\infty \left[C_{m_g}(\infty)-C_{m_g}(\varphi)\right] e^{-i\lambda\varphi}d\varphi \\
f_1(i\lambda) &= \int_0^\infty \left[C_{L_\alpha}(\infty)-C_{L_\alpha}(\varphi)\right] e^{-i\lambda\varphi}d\varphi \\
f_3(i\lambda) &= \int_0^\infty \left[C_{m_\alpha}(\infty)-C_{m_\alpha}(\varphi)\right] e^{-i\lambda\varphi}d\varphi \\
f_2(i\lambda) &= \int_0^\infty \left[C_{L_q}(\infty)-C_{L_q}(\varphi)\right] e^{-i\lambda\varphi}d\varphi \\
f_4(i\lambda) &= \int_0^\infty \left[C_{m_q}(\infty)-C_{m_q}(\varphi)\right] e^{-i\lambda\varphi}d\varphi
\end{aligned}\right\} \qquad (A1)$$

The transforms due to pitching velocity $f_2(i\lambda)$, $f_4(i\lambda)$ will be presented for a wing pitching about the leading edge or the apex. Likewise, all pitching-moment transforms are referred to an axis through the leading edge or the apex. To calculate the transforms for an axis location other than the nose, the following transfer relations may be used:

$$\left.\begin{aligned}
g_1(i\lambda) &= g_{1_O}(i\lambda) \\
g_3(i\lambda) &= g_{3_O}(i\lambda)+\xi g_1(i\lambda) \\
f_1(i\lambda) &= f_{1_O}(i\lambda) \\
f_3(i\lambda) &= f_{3_O}(i\lambda)+\xi f_1(i\lambda) \\
f_2(i\lambda) &= f_{2_O}(i\lambda)-\xi f_1(i\lambda) \\
f_4(i\lambda) &= f_{4_O}(i\lambda)+\xi f_{2_O}(i\lambda)-\xi f_{3_O}(i\lambda)-\xi^2 f_1(i\lambda)
\end{aligned}\right\} \tag{A2}$$

Here, ξ is the number of chord lengths of the new axis from the nose, measured positive rearward from the nose; the subscripted terms are those listed below, referred to the axis $\xi = 0$.

TWO-DIMENSIONAL WING

Transforms

$$g_1(i\lambda) = \frac{4i}{\beta\lambda}\left[f_o(M,\overline{\omega})-1\right] \tag{A3}$$

$$g_{3_O}(i\lambda) = \frac{2i}{\beta\lambda}\left[1-2f_1(M,\overline{\omega})\right] \tag{A4}$$

$$f_1(i\lambda) = \frac{4}{\beta\lambda}\left\{\lambda\left[f_1(M,\overline{\omega})-f_o(M,\overline{\omega})\right]+i\left[f_o(M,\overline{\omega})-1\right]\right\} \tag{A5}$$

$$f_{3_O}(i\lambda) = \frac{2}{\beta\lambda}\left\{\lambda\left[f_o(M,\overline{\omega})-f_2(M,\overline{\omega})\right]+i\left[1-2f_1(M,\overline{\omega})\right]\right\} \tag{A6}$$

$$f_{2_O}(i\lambda) = \frac{2}{\beta\lambda}\left\{\lambda\left[2f_1(M,\overline{\omega})-f_2(M,\overline{\omega})-f_o(M,\overline{\omega})\right]-i\left[1+2f_1(M,\overline{\omega})-2f_o(M,\overline{\omega})\right]\right\} \tag{A7}$$

$$f_{4_0}(i\lambda) = \frac{4}{3\beta\lambda}\left\{\lambda f_2(M,\overline{\omega}) + \left(\frac{\beta^2}{M^2\lambda} - \lambda\right)\left[2f_1(M,\overline{\omega}) - f_0(M,\overline{\omega})\right] + i\left[1 - f_2(M,\overline{\omega}) + 3f_1(M,\overline{\omega}) - 2f_0(M,\overline{\omega})\right]\right\} \tag{A8}$$

In the above, the $f_n(M,\overline{\omega})$ are Schwarz functions (ref. 37), defined as

$$\left.\begin{aligned} f_n(M,\overline{\omega}) &= \int_0^1 u^n e^{-i\overline{\omega}u} J_0\left(\frac{\overline{\omega}u}{M}\right) du \\ \overline{\omega} &= \frac{\lambda M^2}{\beta^2} \end{aligned}\right\} \tag{A9}$$

Numerical tabulations of these functions for $M \geqq 1.20$ may be found in reference 38.

Initial Values

As an aid in plotting the transforms as $\lambda \rightarrow 0$, we give below their first-order expansions.

$$\lim_{\lambda \to 0} g_1(i\lambda) = \frac{2M^2}{\beta^3} - \frac{i\overline{\omega}}{3\beta^3}\left(2M^2+1\right) \tag{A10}$$

$$\lim_{\lambda \to 0} g_{3_0}(i\lambda) = -\frac{4}{3}\frac{M^2}{\beta^3} + \frac{i\overline{\omega}}{4\beta^3}\left(2M^2+1\right) \tag{A11}$$

$$\lim_{\lambda \to 0} f_1(i\lambda) = \frac{2}{\beta^3} - \frac{i\overline{\omega}}{\beta^3} \tag{A12}$$

$$\lim_{\lambda \to 0} f_{3_0}(i\lambda) = -\frac{4}{3\beta^3} + \frac{3}{4}\frac{i\overline{\omega}}{\beta^3} \tag{A13}$$

$$\lim_{\lambda \to 0} f_{2_0}(i\lambda) = \frac{2}{3\beta^3} - \frac{i\overline{\omega}}{4\beta^3} \tag{A14}$$

$$\lim_{\lambda \to 0} f_{4_0}(i\lambda) = -\frac{1}{2\beta^3} + \frac{i\bar{\omega}}{5\beta^3} \tag{A15}$$

Stability Derivatives

Finally, in order to complete the compilation of terms needed to calculate the harmonic responses, equations (13) and (14) in the text, the necessary stability derivatives are listed below.

$$C_{L_\alpha}(\infty) = C_{L_g}(\infty) = \frac{4}{\beta} \tag{A16}$$

$$C_{m_{\alpha_0}}(\infty) = C_{m_{g_0}}(\infty) = -\frac{2}{\beta} \tag{A17}$$

$$C_{L_{\dot{\alpha}}} = -\frac{2}{\beta^3} \tag{A18}$$

$$C_{m_{\dot{\alpha}_0}} = \frac{4}{3\beta^3} \tag{A19}$$

$$C_{L_{q_0}}(\infty) = \frac{2}{\beta} \tag{A20}$$

$$C_{m_{q_0}}(\infty) = -\frac{4}{3\beta} \tag{A21}$$

RECTANGULAR WING

The harmonic functions and stability derivatives for the rectangular wing are each composed of a two-dimensional contribution plus a contribution dependent on aspect ratio. Listed below are the aspect-ratio contributions, each of which should be added to the corresponding two-dimensional function, already given. The results are valid only for those combinations of Mach number and aspect ratio for which $\beta A \gtrsim 1$.

Transforms

$$\Delta g_1(i\lambda) = \frac{2i}{MA\lambda^3}\left(\frac{2}{M} + \frac{M\lambda^2}{\beta^2} + me^{-i\lambda/m} - ne^{-i\lambda/n}\right) \tag{A22}$$

$$\Delta g_{3_0}(i\lambda) = -\frac{1}{MA\lambda^3}\left[\frac{8}{M\lambda} + \frac{4}{3}\, i\, \frac{M\lambda^2}{\beta^2} + 2i\left(me^{-i\lambda/m} - ne^{-i\lambda/n}\right) + \frac{2}{\lambda}\left(m^2e^{-i\lambda/m} - n^2e^{-i\lambda/n}\right)\right] \tag{A23}$$

$$\Delta f_1(i\lambda) = \frac{2}{M^2A\lambda^2}\left[-2 + i\left(\frac{\lambda M^2}{\beta^2} - \frac{2}{\lambda}\right) + \frac{i}{\lambda}\left(me^{-i\lambda/m} + ne^{-i\lambda/n}\right)\right] \tag{A24}$$

$$\Delta f_{3_0}(i\lambda) = \frac{2}{M^2A\lambda^2}\left[1 + \frac{2(M^2+1)}{M^2\lambda^2} - i\,\frac{2M^2\lambda}{3\beta^2} - \frac{1}{\lambda^2}\left(m^2e^{-i\lambda/m} + n^2e^{-i\lambda/n}\right) - \frac{i}{\lambda}\left(me^{-i\lambda/m} + ne^{-i\lambda/n}\right)\right] \tag{A25}$$

$$\Delta f_{2_0}(i\lambda) = \frac{2}{M^2A\lambda^2}\left[-1 + \frac{2(M^2+1)}{M^2\lambda^2} + i\left(\frac{M^2\lambda}{3\beta^2} - \frac{2}{\lambda}\right) - \frac{1}{\lambda^2}\left(m^2e^{-i\lambda/m} + n^2e^{-i\lambda/n}\right)\right] \tag{A26}$$

$$\Delta f_{4_0}(i\lambda) = \frac{2}{3M^2A\lambda^2}\left\{2 + \frac{3i}{\lambda}\left[1 - \frac{M^2\lambda^2}{4\beta^2} + \frac{2(M^2+3)}{M^2\lambda^2}\right] + \frac{3}{\lambda^2}\left(m^2e^{-i\lambda/m} + n^2e^{-i\lambda/n}\right) - \frac{3i}{\lambda^3}\left(m^3e^{-i\lambda/m} + n^3e^{-i\lambda/n}\right)\right\} \tag{A27}$$

where

$$n = \frac{M+1}{M}$$

$$m = \frac{M-1}{M}$$

Initial Values

$$\lim_{\lambda\to 0} \Delta g_1(i\lambda) = -\frac{4}{3}\,\frac{M^2}{\beta^4 A} + \frac{i\overline{\omega}}{6\beta^4 A}\,(3M^2+1) \tag{A28}$$

$$\lim_{\lambda \to 0} \Delta g_{3_0}(i\lambda) = \frac{M^2}{\beta^4 A} - \frac{2}{15}\frac{i\bar{\omega}}{\beta^4 A}(3M^2+1) \tag{A29}$$

$$\lim_{\lambda \to 0} \Delta f_1(i\lambda) = -\frac{2}{3\beta^4 A}(M^2+1) + \frac{i\bar{\omega}}{6\beta^4 A}(M^2+3) \tag{A30}$$

$$\lim_{\lambda \to 0} \Delta f_{3_0}(i\lambda) = \frac{1}{2\beta^4 A}(M^2+1) - \frac{2}{15}\frac{i\bar{\omega}}{\beta^4 A}(M^2+3) \tag{A31}$$

$$\lim_{\lambda \to 0} \Delta f_{2_0}(i\lambda) = -\frac{1}{6\beta^4 A}(M^2+1) + \frac{1}{30}\frac{i\bar{\omega}}{\beta^4 A}(M^2+3) \tag{A32}$$

$$\lim_{\lambda \to 0} \Delta f_{4_0}(i\lambda) = \frac{2}{15}\frac{(M^2+1)}{\beta^4 A} - \frac{1}{36}\frac{i\bar{\omega}}{\beta^4 A}(M^2+3) \tag{A33}$$

Stability Derivatives

$$\Delta C_{L_\alpha}(\infty) = \Delta C_{L_g}(\infty) = -\frac{2}{\beta^2 A} \tag{A34}$$

$$\Delta C_{m_{\alpha_0}}(\infty) = \Delta C_{m_{g_0}}(\infty) = \frac{4}{3\beta^2 A} \tag{A35}$$

$$\Delta C_{L_{\dot{\alpha}}} = \frac{2}{3\beta^4 A}(M^2+1) \tag{A36}$$

$$\Delta C_{m_{\dot{\alpha}_0}} = -\frac{1}{2\beta^4 A}(M^2+1) \tag{A37}$$

$$\Delta C_{L_{q_0}}(\infty) = -\frac{2}{3\beta^2 A} \tag{A38}$$

$$\Delta C_{m_{q_0}}(\infty) = \frac{1}{2\beta^2 A} \tag{A39}$$

WIDE TRIANGULAR WING

Presented below are the transforms, their initial values, and pertinent stability derivatives for the wide triangular wing. Results are applicable so long as the wing leading edge is supersonic; hence it is a requirement that $\beta A \geq 4$.

Transforms

$$g_1(i\lambda) = \frac{4}{\beta\lambda}\left\{\frac{2}{\lambda}\left[f_o(M,\overline{\omega}) - e^{-i\lambda}f_o\left(\frac{1}{M}, \frac{\overline{\omega}}{M^2}\right)\right] - i\right\} \tag{A40}$$

$$g_{3_o}(i\lambda) = \frac{8}{3\beta\lambda}\left\{\frac{3}{\lambda}\left[e^{-i\lambda}f_o\left(\frac{1}{M}, \frac{\overline{\omega}}{M^2}\right) - f_1(M,\overline{\omega})\right] + i\left[1 + \frac{3}{\lambda^2}\left(f_o(M,\overline{\omega}) - e^{-i\lambda}f_o\left(\frac{1}{M}, \frac{\overline{\omega}}{M^2}\right)\right)\right]\right\} \tag{A41}$$

$$f_1(i\lambda) = \frac{4}{\beta\lambda}\left\{\lambda\left[2f_1(M,\overline{\omega}) - f_2(M,\overline{\omega}) - f_o(M,\overline{\omega})\right] + i\left[2f_o(M,\overline{\omega}) - 2f_1(M,\overline{\omega}) - 1\right]\right\} \tag{A42}$$

$$f_{3_o}(i\lambda) = \frac{8}{3\beta\lambda}\left\{\lambda f_2(M,\overline{\omega}) + \left(\lambda - \frac{1}{\overline{\omega}}\right)\left[f_o(M,\overline{\omega}) - 2f_1(M,\overline{\omega})\right] + i\left[1 - 2f_o(M,\overline{\omega}) + 3f_1(M,\overline{\omega}) - f_2(M,\overline{\omega})\right]\right\} \tag{A43}$$

$$f_{2_o}(i\lambda) = \frac{8}{3\beta\lambda}\left\{-\lambda f_2(M,\overline{\omega}) + \left(\lambda + \frac{2}{\overline{\omega}}\right)\left[2f_1(M,\overline{\omega}) - f_o(M,\overline{\omega})\right] + i\left[-1 - 2f_2(M,\overline{\omega}) + 2f_o(M,\overline{\omega})\right]\right\} \tag{A44}$$

$$f_{4_o}(i\lambda) = \frac{2}{\beta\lambda}\left\{\lambda f_2(M,\overline{\omega})\left(1 + \frac{2M^2+3}{M^2\lambda^2}\right) - 2\left(\lambda + \frac{3}{\lambda}\right)f_1(M,\overline{\omega}) + \left(1 + \frac{2M^2-1}{M^2\lambda^2}\right)\lambda f_o(M,\overline{\omega}) + i\left[1 + 2f_2(M,\overline{\omega}) + \frac{4}{\overline{\omega}\lambda}f_1(M,\overline{\omega}) - 2\left(1 + \frac{1}{\lambda\overline{\omega}}\right)f_o(M,\overline{\omega})\right]\right\} \tag{A45}$$

The $f_o\left(\frac{1}{M}, \frac{\overline{\omega}}{M^2}\right)$ function which appears in $g_1(i\lambda)$ and $g_3(i\lambda)$ (eqs. (A40) and (A41)) is defined as,

$$f_o\left(\frac{1}{M}, \frac{\overline{\omega}}{M^2}\right) = \int_0^1 e^{-\frac{i\overline{\omega}}{M^2}u} J_o\left(\frac{\overline{\omega}u}{M}\right) du \tag{A46}$$

Tabulations of this function are not as yet available; in their absence the function can be evaluated either by numerical integration or by means of the series expansion, (cf. ref. 37)

$$f_o\left(\frac{1}{M}, \frac{\overline{\omega}}{M^2}\right) = e^{-\frac{i\overline{\omega}}{M^2}} \sum_{n=0}^{\infty} \frac{1}{2^n n!(2n+1)} \left(-\frac{\beta^2\overline{\omega}}{M^2}\right)^n \left[J_n\left(\frac{\overline{\omega}}{M^2}\right) + iJ_{n+1}\left(\frac{\overline{\omega}}{M^2}\right)\right] \tag{A47}$$

Initial Values

$$\lim_{\lambda \to 0} g_1(i\lambda) = \frac{4}{3\beta^3}(2M^2-1) - \frac{i\overline{\omega}}{6M^2\beta^3}(6M^4-5M^2+2) \tag{A48}$$

$$\lim_{\lambda \to 0} g_{3_o}(i\lambda) = -\frac{1}{\beta^3}(2M^2-1) + \frac{2}{15}\frac{i\overline{\omega}}{M^2\beta^3}(6M^4-5M^2+2) \tag{A49}$$

$$\lim_{\lambda \to 0} f_1(i\lambda) = \frac{4}{3\beta^3} - \frac{i\overline{\omega}}{2\beta^3} \tag{A50}$$

$$\lim_{\lambda \to 0} f_{3_o}(i\lambda) = -\frac{1}{\beta^3} + \frac{2}{5}\frac{i\overline{\omega}}{\beta^3} \tag{A51}$$

$$\lim_{\lambda \to 0} f_{2_o}(i\lambda) = \frac{2}{3\beta^3} - \frac{i\overline{\omega}}{5\beta^3} \tag{A52}$$

$$\lim_{\lambda \to 0} f_{4_o}(i\lambda) = -\frac{8}{15\beta^3} + \frac{i\overline{\omega}}{6\beta^3} \tag{A53}$$

Stability Derivatives

$$C_{L_\alpha}(\infty) = C_{L_g}(\infty) = \frac{4}{\beta} \tag{A54}$$

$$C_{m_{\alpha_o}}(\infty) = C_{m_{g_o}}(\infty) = -\frac{8}{3\beta} \tag{A55}$$

$$C_{L_{\dot\alpha}} = -\frac{4}{3\beta^3} \tag{A56}$$

$$C_{m_{\dot\alpha_o}} = \frac{1}{\beta^3} \tag{A57}$$

$$C_{L_{q_o}}(\infty) = \frac{8}{3\beta} \tag{A58}$$

$$C_{m_{q_o}}(\infty) = -\frac{2}{\beta} \tag{A59}$$

APPENDIX B

SOLUTION OF INTEGRAL EQUATION FOR $P(\varphi)$

The following is a compilation of expressions which arise in solving the integral equation for $P(\varphi)$, equation (47) of the text. The equation is readily solved by the use of the two-sided Laplace transform (ref. 39). For any function $F(\varphi)$, define the transform as

$$f(s) = \int_{-\infty}^{\infty} F(\varphi)\, e^{-s\varphi} d\varphi \tag{B1}$$

and we denote the transforms that result from integration of $F(\varphi)$ for positive and negative values of φ by $f_+(s)$ and $f_-(s)$, respectively. Thus

$$\left.\begin{aligned} f_+(s) &= \int_0^{\infty} e^{-s\varphi} F(\varphi)\, d\varphi \\ f_-(s) &= \int_{-\infty}^{0} e^{-s\varphi} F(\varphi)\, d\varphi \end{aligned}\right\} \tag{B2}$$

The solution may be put in the form

$$p_+(s) = \frac{-\int_0^{\infty} \alpha(\varphi)\, e^{-s\varphi} d\varphi}{\theta_+(s)} \tag{B3}$$

where $p_+(s)$ is the transform of the desired quantity $P(\varphi)$. The latter quantity is different from zero only in the range $\varphi > 0$; it is zero for $\varphi < 0$, as it must be to be physically realizable as an impulse response.

The following quantities arise in the derivation of $\alpha(\varphi)$ and $\theta_+(s)$:

<u>$\sigma_+(s)$, $S(\varphi)$:</u> From equations (27), (51), and (53)

$$\sigma_+(s) = \frac{\lambda_2 s^2 + \lambda_1 s}{(s+a_1)(s+a_1^*)(s+p)^3} \tag{B4}$$

where

$$a_1 = \mu_1 + i\sqrt{k_1^2-\mu_1^2}$$

$$\mu_1 = -\frac{1}{2\zeta}\left[\left(C_{m_q}+C_{m_{\dot{\alpha}}}\right) + x_o\left(C_{L_q}+C_{L_{\dot{\alpha}}}\right)\right]$$

$$k_1^2 = -\frac{1}{\zeta}\left(C_{m_\alpha}+x_oC_{L_\alpha}\right)$$

$$\lambda_2 = \frac{1}{\zeta}\left[p^2\left(C_{m_\alpha}+x_oC_{L_\alpha}\right) - 2rC_{m_\alpha}\right]$$

$$\lambda_1 = \frac{p^3}{\zeta}\left(C_{m_\alpha}+x_oC_{L_\alpha}\right)$$

The inversion of $\sigma_+(s)$ gives

$$S(\varphi) = (s_o+s_1\varphi+s_2\varphi^2)e^{-p\varphi} + s_3e^{-a_1\varphi} + s_3{}^*e^{-a_1{}^*\varphi} \tag{B5}$$

where

$$s_2 = \frac{1}{2}\left[\frac{\lambda_2s^2+\lambda_1s}{(s+a_1)(s+a_1{}^*)}\right]_{s=-p}$$

$$s_1 = \frac{d}{ds}\left[\frac{\lambda_2s^2+\lambda_1s}{(s+a_1)(s+a_1{}^*)}\right]_{s=-p}$$

$$s_o = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{\lambda_2s^2+\lambda_1s}{(s+a_1)(s+a_1{}^*)}\right]_{s=-p}$$

$$s_3 = \left[\frac{\lambda_2s^2+\lambda_1s}{(s+p)^3(s+a_1{}^*)}\right]_{s=-a_1}$$

$$s_3{}^* = \left[\frac{\lambda_2s^2+\lambda_1s}{(s+p)^3(s+a_1)}\right]_{s=-a_1{}^*}$$

$\underline{\rho_+(s)}$: From equations (27) and (51),

$$\rho_+(s) = \frac{1}{K}\frac{s+c_1}{(s+a_1)(s+a_1^*)} \tag{B6}$$

where

$$K = -\frac{\zeta}{\eta' x_o + C_{m_{\dot{\alpha}}}}$$

$$c_1 = \frac{x_o C_{L_\alpha} + C_{m_\alpha}}{\eta' x_o + C_{m_{\dot{\alpha}}}}$$

$\underline{w(s)}$: Using equation (57)

$$w(s) = \int_{-\infty}^{\infty} W(\varphi)e^{-s\varphi}d\varphi$$

$$= 3N\overline{\alpha_g{}^2}\,\frac{\gamma^2 - s^2}{(N^2 - s^2)^2} \tag{B7}$$

where

$$N = \frac{c}{L}$$

$$\gamma^2 = \frac{N^2}{3}$$

$\underline{u(s), U(\varphi)}$: Using equation (42)

$$u(s) = \int_{-\infty}^{\infty} U(\varphi)e^{-s\varphi}d\varphi = \sigma_+(s)w(s) \tag{B8}$$

To find $U(\varphi)$ for $\varphi > 0$, we evaluate the residues of the product of $\sigma_+(s)$ and $w(s)$ at poles in the left-half plane. The result is

$$U(\varphi)(\varphi > 0) = (U_o + U_1\varphi + U_2\varphi^2)e^{-p\varphi} + U_3e^{-a_1\varphi} +$$

$$U_3{}^*e^{-a_1{}^*\varphi} + (U_4 + U_5\varphi)e^{-N\varphi} \tag{B9}$$

where

$$U_0 = s_2w''(-p) + s_1w'(-p) + s_0w(-p)$$

$$U_1 = 2s_2w'(-p) + s_1w(-p)$$

$$U_2 = s_2w(-p)$$

$$U_3 = s_3w(-a_1)$$

$$U_3^* = s_3^*w(-a_1^*)$$

$$U_4 = \overline{\alpha_g^2}\left[\sigma_+(-N) - \frac{N}{2}\sigma_+'(-N)\right]$$

$$U_5 = -\overline{\alpha_g^2}\,\frac{N}{2}\,\sigma_+(-N)$$

$\underline{\psi(s)}$: The two-sided Laplace transform of the autocorrelation function $\overline{\Psi(\varphi)}$ may be written (using eq. (45))

$$\psi(s) = \int_{-\infty}^{\infty} \Psi(\varphi)e^{-s\varphi}d\varphi = w(s)\left[1 + \frac{1}{a\rho_+(s)\rho_+(-s)}\right]$$

$$= \frac{K^2w(s)}{a(c_1^2-s^2)}\,(s^4-2\gamma_2s^2+\gamma_0^2) \qquad \text{(B10)}$$

where

$$2\gamma_2 = a_1^2 + a_1^{*2} + \frac{a}{K^2}$$

$$\gamma_0^2 = (a_1a_1^*)^2 + \frac{ac_1^2}{K^2}$$

This may be factored to the desired form

$$\psi(s) = \theta_+(s)\theta_-(s) \qquad \text{(B11)}$$

where

$$\theta_+(s) = \Xi \frac{(\gamma+s)(b+s)(b^*+s)}{(N+s)^2(c_1+s)}$$

$$\theta_-(s) = \Xi \frac{(\gamma-s)(b-s)(b^*-s)}{(N-s)^2(c_1-s)}$$

and

$$\Xi = \left(\frac{3NK^2\overline{\alpha_g{}^2}}{a}\right)^{\frac{1}{2}}$$

$$b = \beta + i\alpha$$

$$\beta = \sqrt{\frac{|\gamma_0|+\gamma_2}{2}}$$

$$\alpha = \sqrt{\frac{|\gamma_0|-\gamma_2}{2}}$$

$\underline{\alpha(\varphi)}$: Letting

$$A(s) = \frac{u(s)}{\theta_-(s)} \tag{B12}$$

we find $\alpha(\varphi)$ for $\varphi > 0$ by evaluating the residues of $A(s)$ at poles in the left-half plane. Let

$$\Delta_-(s) = \frac{1}{\theta_-(s)} \tag{B13}$$

Then

$$\alpha(\varphi)(\varphi > 0) = (A_0+A_1\varphi+A_2\varphi^2)e^{-p\varphi} + A_3e^{-a_1\varphi} + A_3{}^*e^{-a_1{}^*\varphi} + (A_4+A_5\varphi)e^{-N\varphi} \tag{B14}$$

where

$$A_0 = U_0\Delta_-(-p) + U_1\Delta_-'(-p) + U_2\Delta_-''(-p)$$

$$A_1 = U_1\Delta_-(-p) + 2U_2\Delta_-'(-p)$$

$$A_2 = U_2\Delta_-(-p)$$

$$A_3 = U_3\Delta_-(-a_1)$$

$$A_3^* = U_3^*\Delta_-(-a_1^*)$$

$$A_4 = U_4\Delta_-(-N) + U_5\Delta_-'(-N)$$

$$A_5 = U_5\Delta_-(-N)$$

$\underline{P(\varphi)}$: Having determined $\theta_+(s)$ (eq. (B11)) and $\alpha(\varphi)$ (eq. (B14)), we find $P(\varphi)$ by inversion of equation (B3). The results may be written

$$P(\varphi) = (P_0+P_1\varphi+P_2\varphi^2)e^{-p\varphi} + P_3e^{-a_1\varphi} + P_3^*e^{-a_1^*\varphi} +$$

$$P_6e^{-\gamma\varphi} + P_7e^{-b\varphi} + P_7^*e^{-b^*\varphi}$$

where

$$P_0 = -U_0\Delta(-p) - U_1\Delta'(-p) - U_2\Delta''(-p)$$

$$P_1 = -U_1\Delta(-p) - 2U_2\Delta'(-p)$$

$$P_2 = -U_2\Delta(-p)$$

$$P_3 = -U_3\Delta(-a_1)$$

$$P_3^* = -U_3^*\Delta(-a_1^*)$$

$$P_6 = -\frac{1}{\Xi}\,\frac{(N-\gamma)^2(c_1-\gamma)}{(b-\gamma)(b^*-\gamma)}\,A_+(-\gamma)$$

$$P_7 = -\frac{1}{\Xi}\,\frac{(N-b)^2(c_1-b)}{(\gamma-b)(b^*-b)}\,A_+(-b)$$

$$P_7^* = -\frac{1}{\Xi}\,\frac{(N-b^*)^2(c_1-b^*)}{(\gamma-b^*)(b-b^*)}\,A_+(-b^*)$$

$$\Delta(s) = \frac{1}{\psi(s)}$$

$$A_+(s) = \frac{A_0}{p+s} + \frac{A_1}{(p+s)^2} + \frac{2A_2}{(p+s)^3} + \frac{A_3}{(a_1+s)} + \frac{A_3^*}{a_1^*+s} + \frac{A_4}{N+s} + \frac{A_5}{(N+s)^2}$$

TABULATION OF CONSTANTS

Given below is a tabulation of constants which were used in numerical computations to describe the airplane's inertial and geometric properties, the system functions, and the gust correlation function.

Airplane:

Mass parameter	$\eta' = 100$
Inertia parameter	$\zeta = 100$
Wing chord	$c = 30$ ft

Center of gravity located at 0.25c

Airplane system functions:

$C_{L_\alpha} = 3.0$	$C_{m_{\dot{\alpha}}} = 0.50$
$C_{m_\alpha} = -0.75$	$\mu = 0.0175$
$C_{L_q} = 2.0$	$k = 0.0861$
$C_{m_q} = -1.0$	$p = 0.20$
$C_{L_{\dot{\alpha}}} = -1.0$	$r = 0.021$

Gust correlation function:

$$L = 300 \text{ ft}$$

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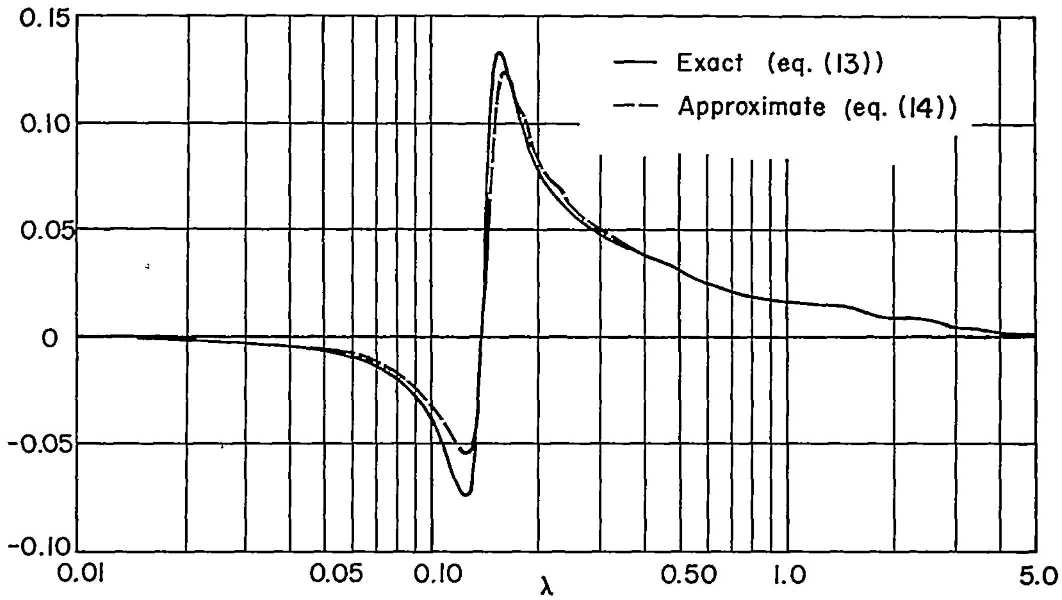
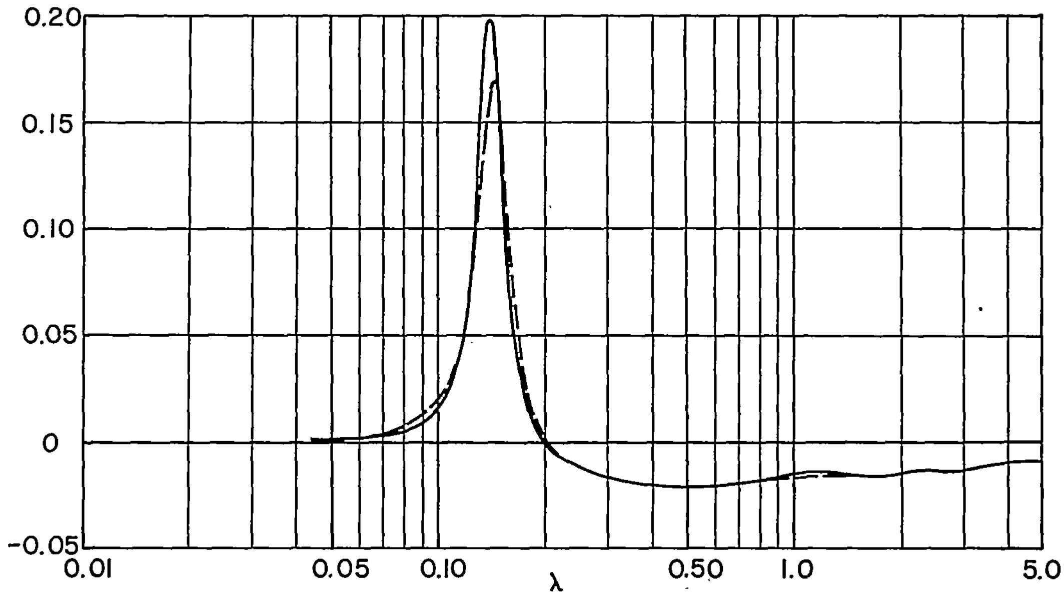
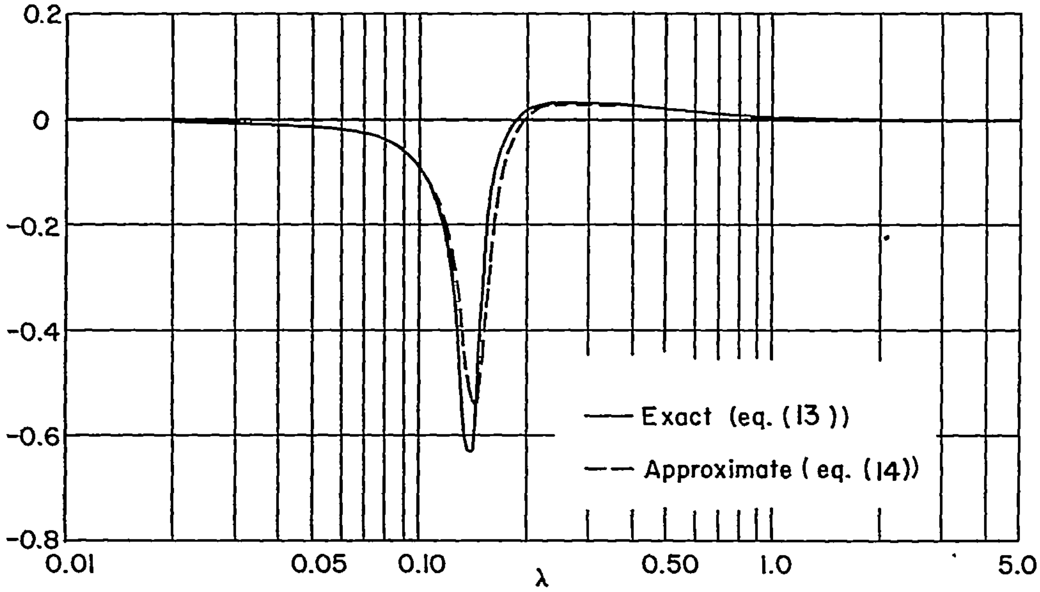


(a) R.P. $\left[\frac{\gamma'}{\alpha_g}(i\lambda)\right]$

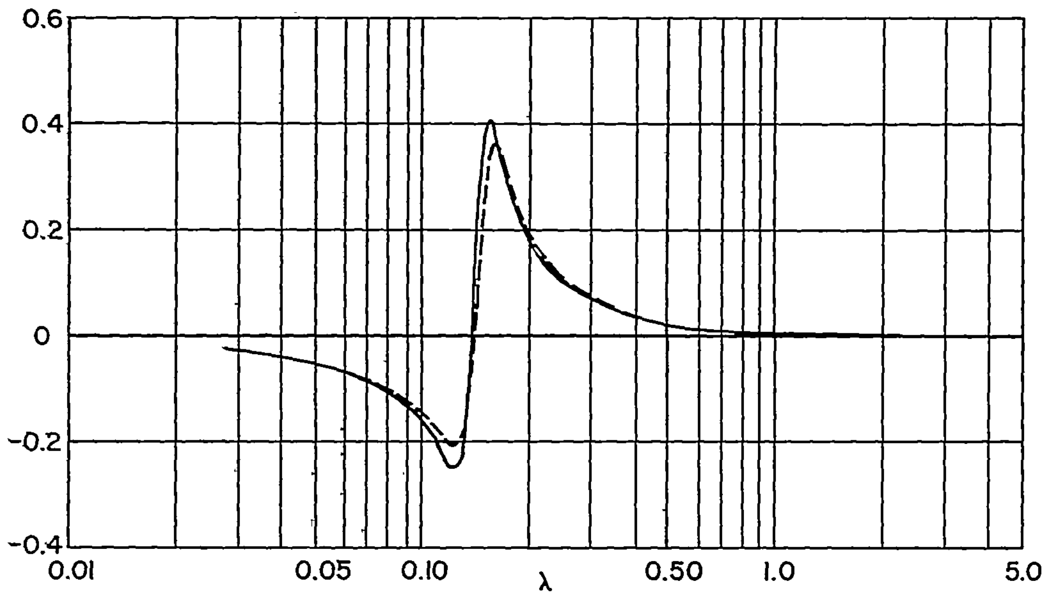


(b) I.P. $\left[\frac{\gamma'}{\alpha_g}(i\lambda)\right]$

Figure 1.- Normal acceleration response to harmonic gusts of an aspect-ratio-3 rectangular wing flying at Mach number 1.2.



(a) R.P. $\left[\frac{q}{\alpha_g}(i\lambda)\right]$



(b) I.P. $\left[\frac{q}{\alpha_g}(i\lambda)\right]$

Figure 2.- Pitching velocity response to harmonic gusts of an aspect-ratio-3 rectangular wing flying at Mach number 1.2.

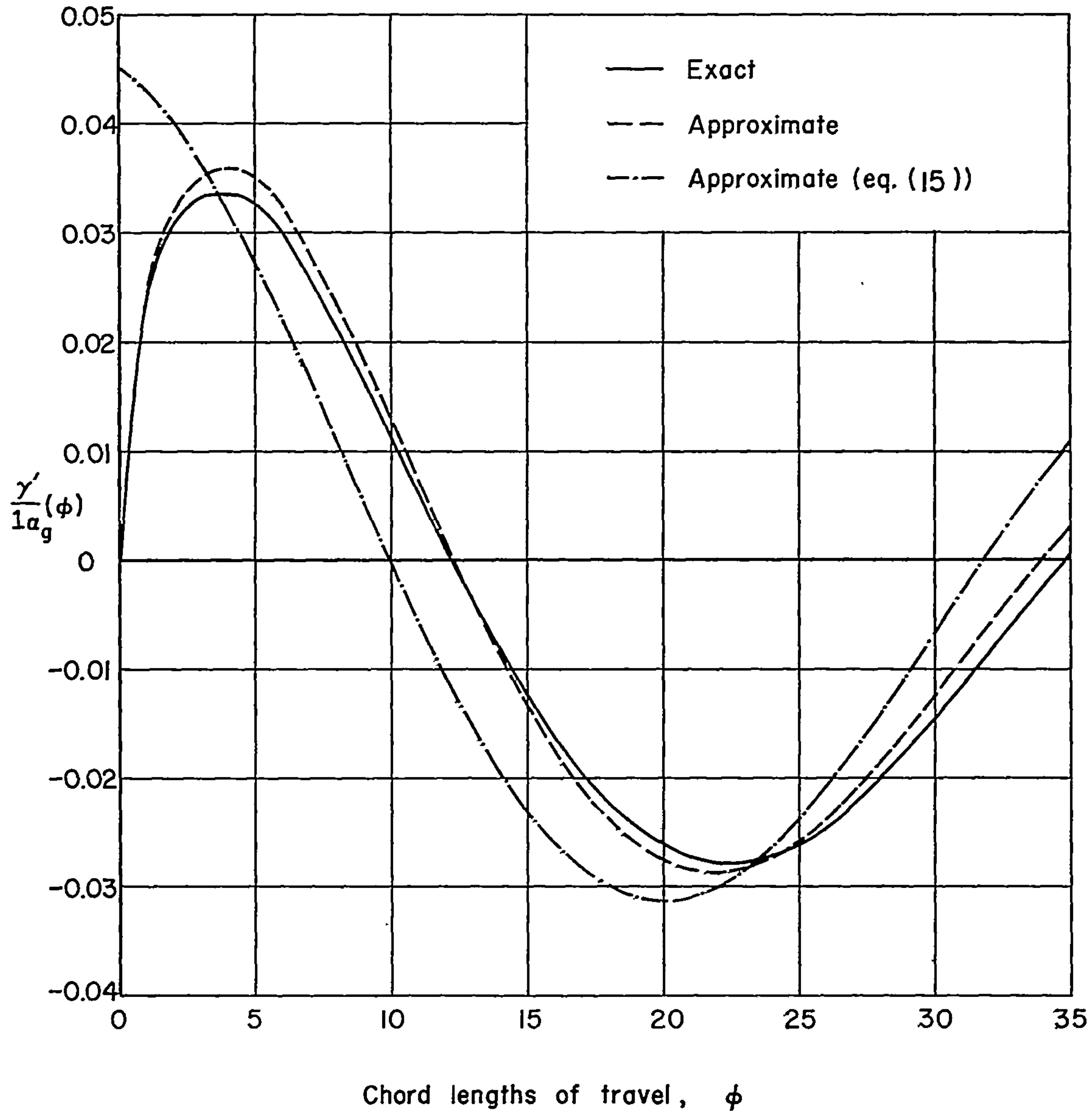


Figure 3.- Normal acceleration response to sharp-edge gust of an aspect-ratio-3 rectangular wing flying at Mach number 1.2.

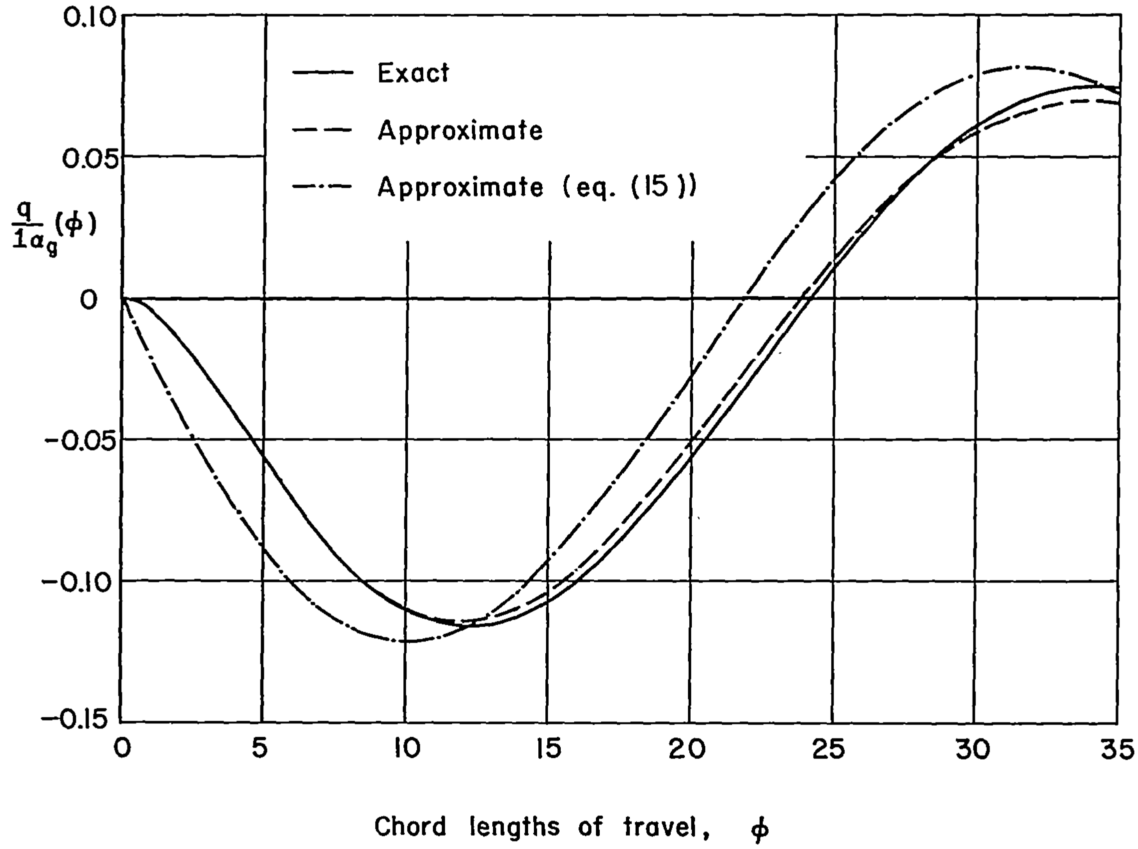
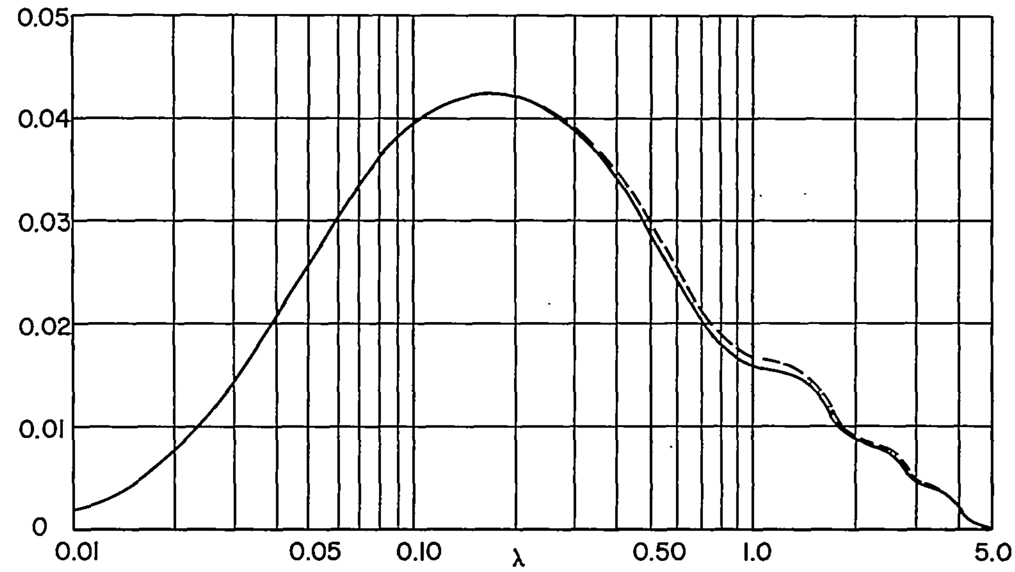
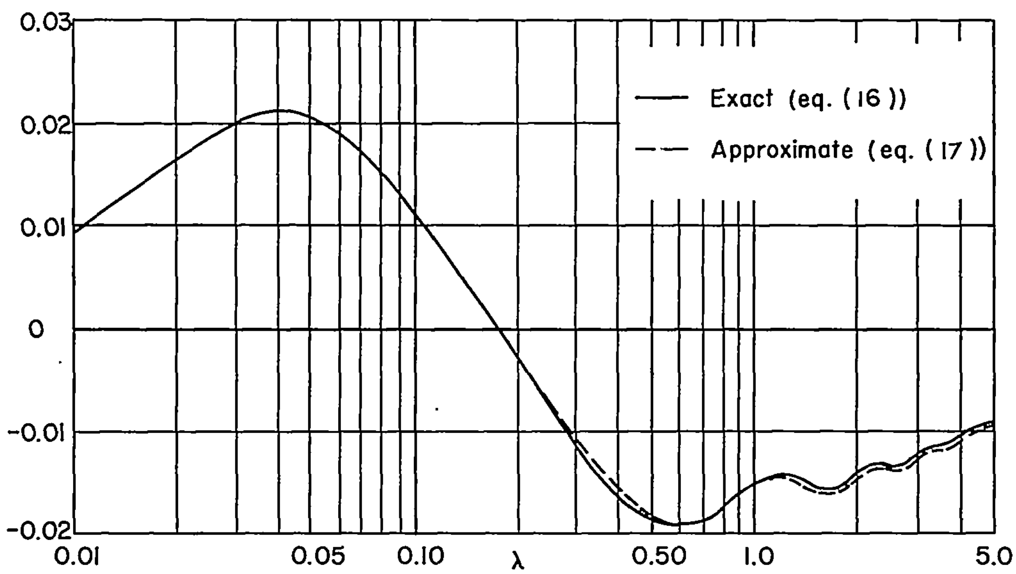


Figure 4.- Pitching velocity response to sharp-edge gust of an aspect-ratio-3 rectangular wing flying at Mach number 1.2.



(a) R.P. $\left[\frac{\gamma'}{\alpha_g}(i\lambda)\right]$



(b) I.P. $\left[\frac{\gamma'}{\alpha_g}(i\lambda)\right]$

Figure 5.- Normal acceleration response to harmonic gusts of an aspect-ratio-3 rectangular wing that is restrained from pitching; Mach number 1.2.

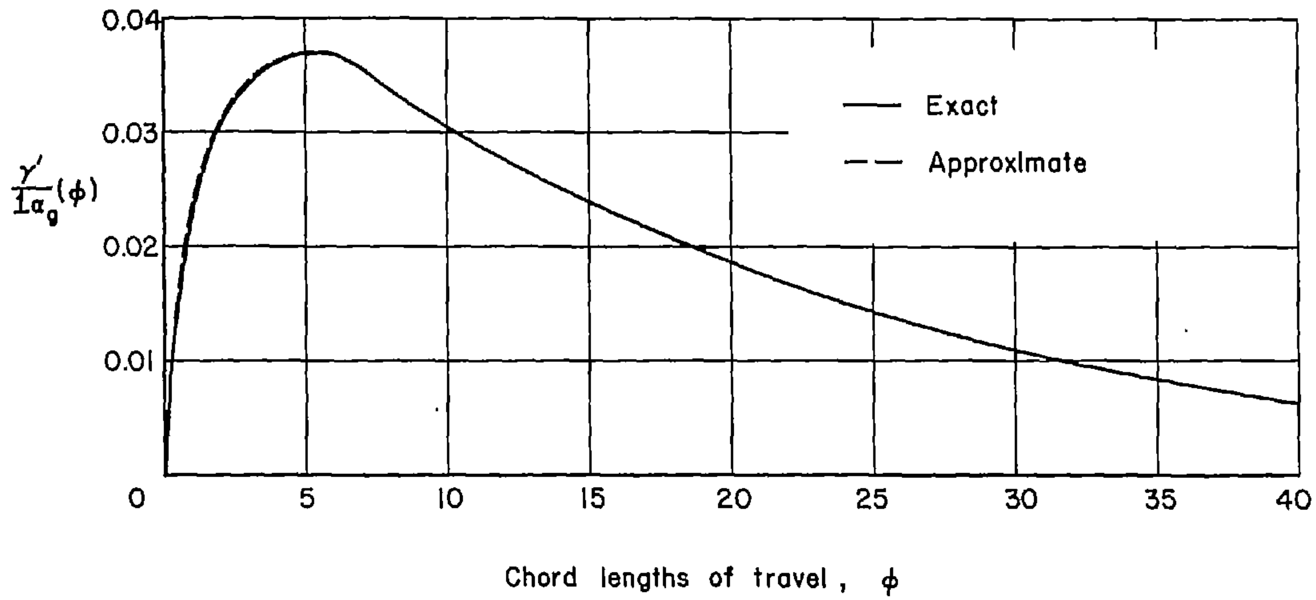


Figure 6.- Normal acceleration response to sharp-edge gust of an aspect-ratio-3 rectangular wing that is restrained from pitching; Mach number 1.2.

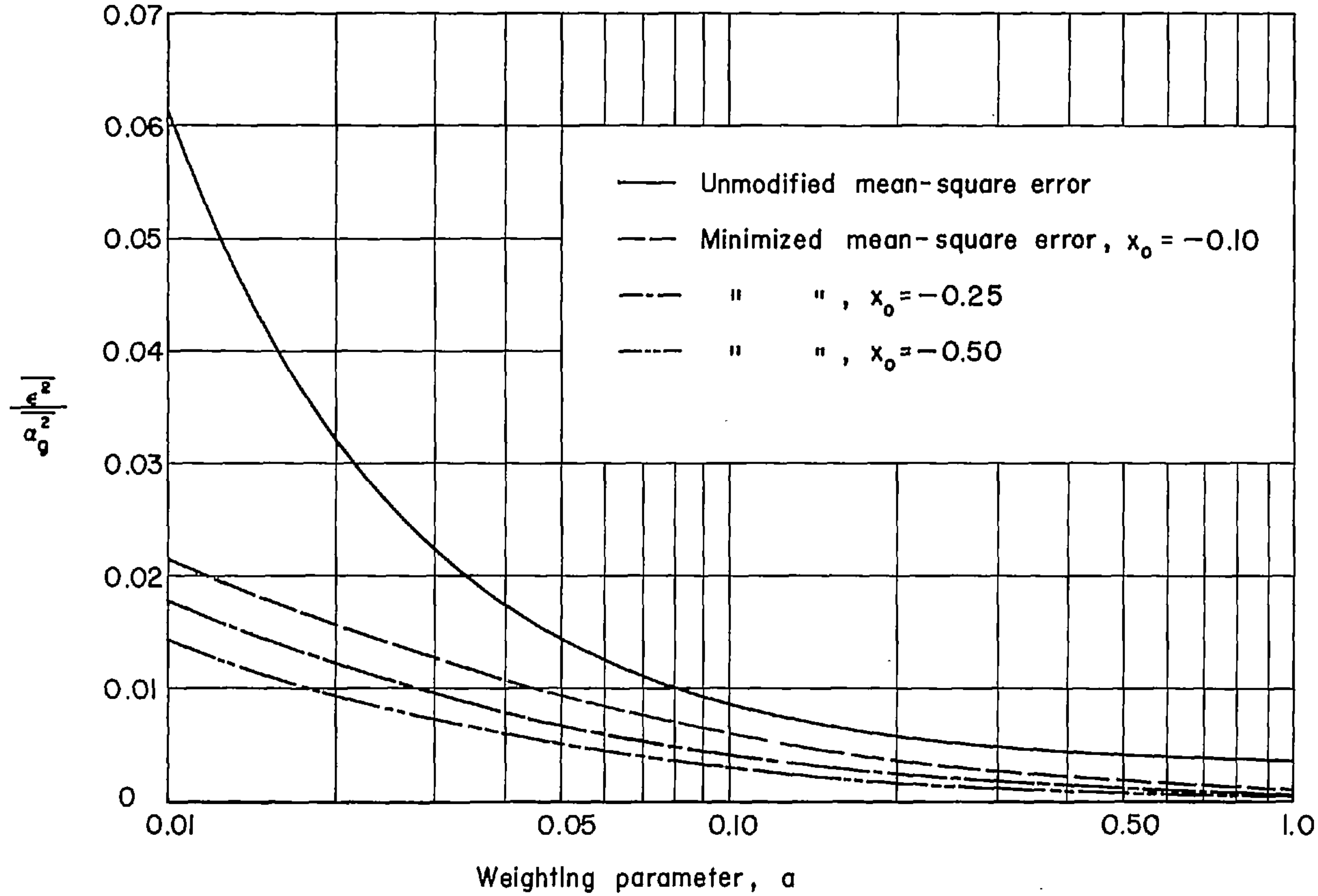


Figure 7.- Comparison of minimized and unmodified mean-square errors for several values of control-force position.

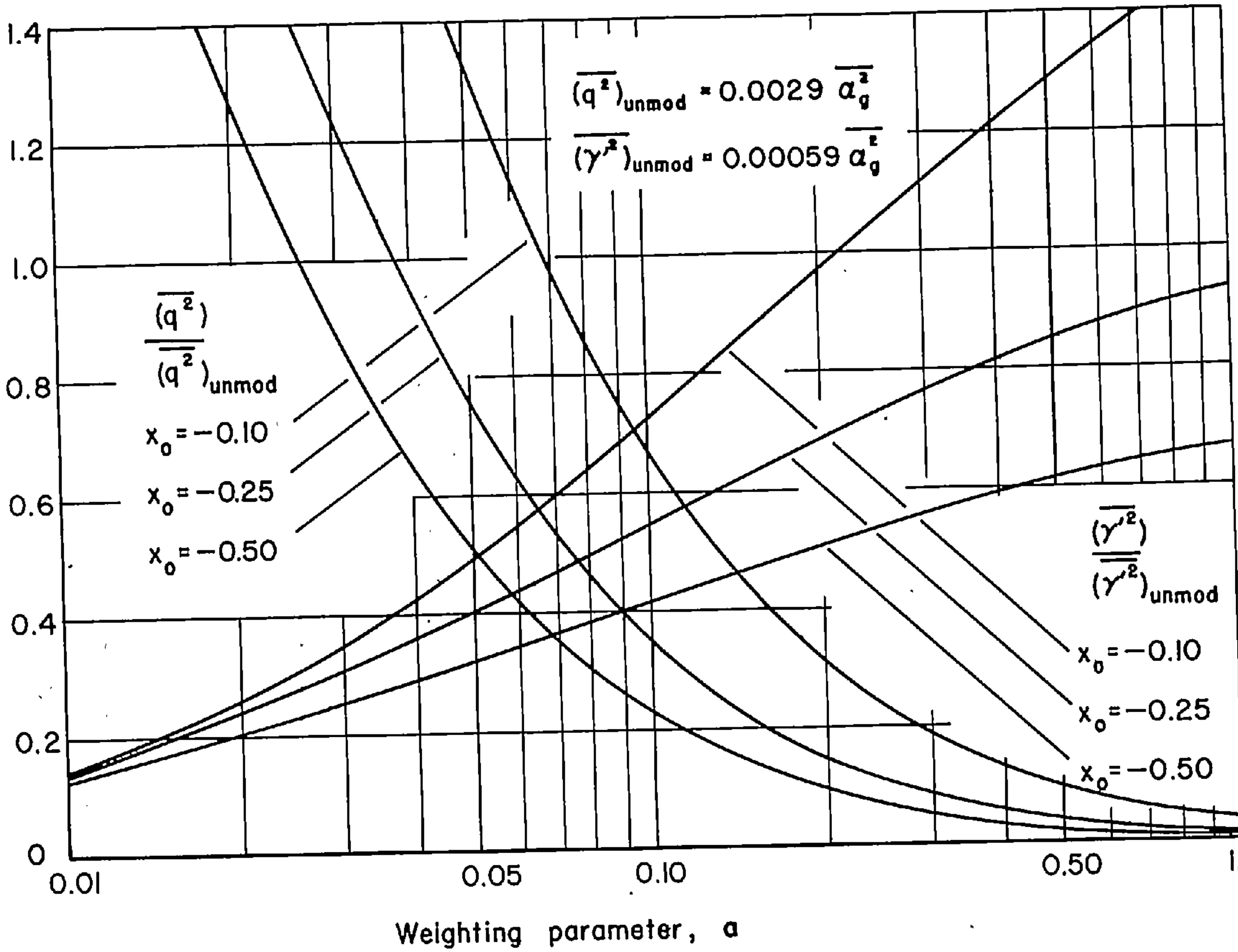


Figure 8.- Ratios of individual components of minimized mean-square error to their corresponding components of unmodified mean-square error.

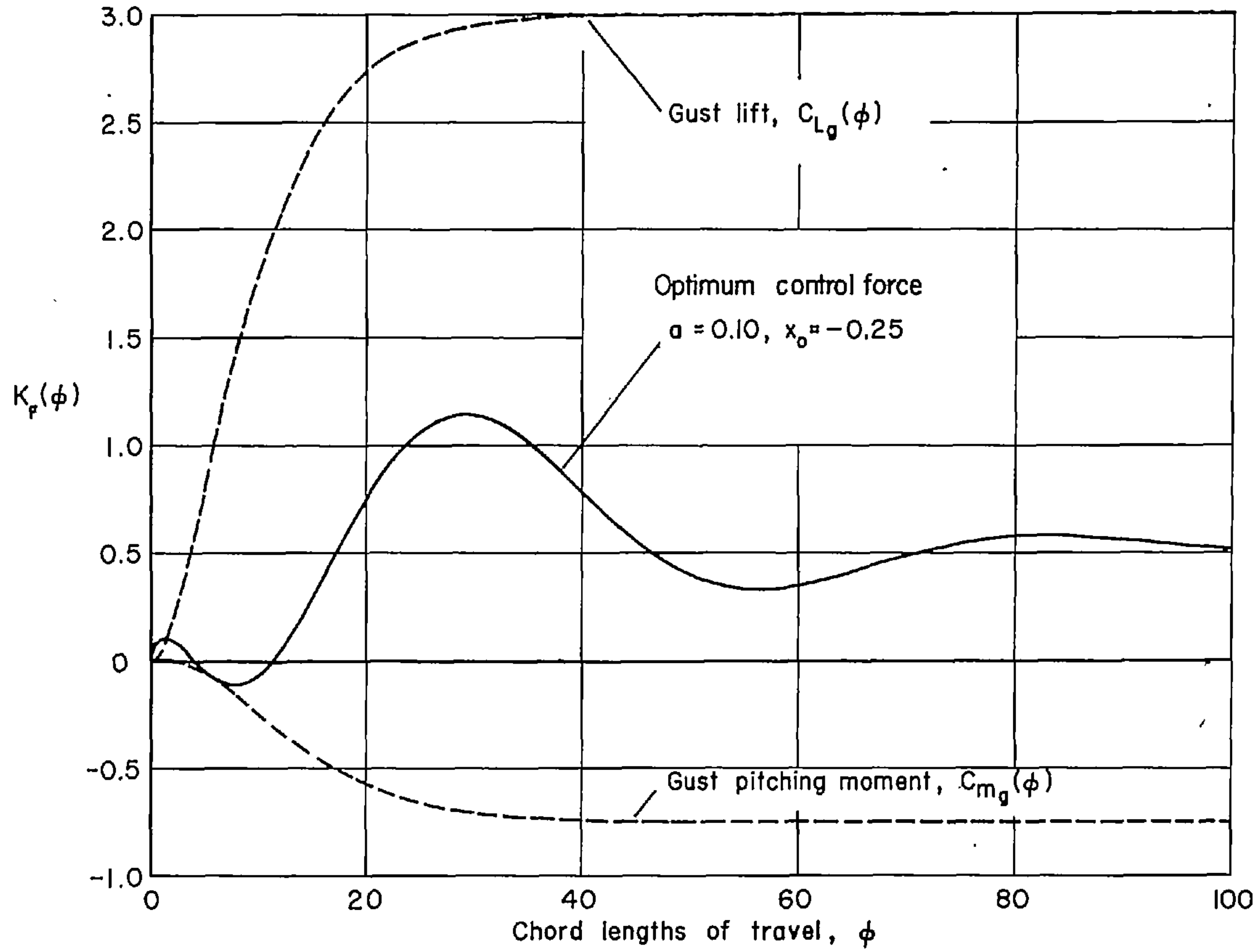


Figure 9.- Control-force response to sharp-edge gust.

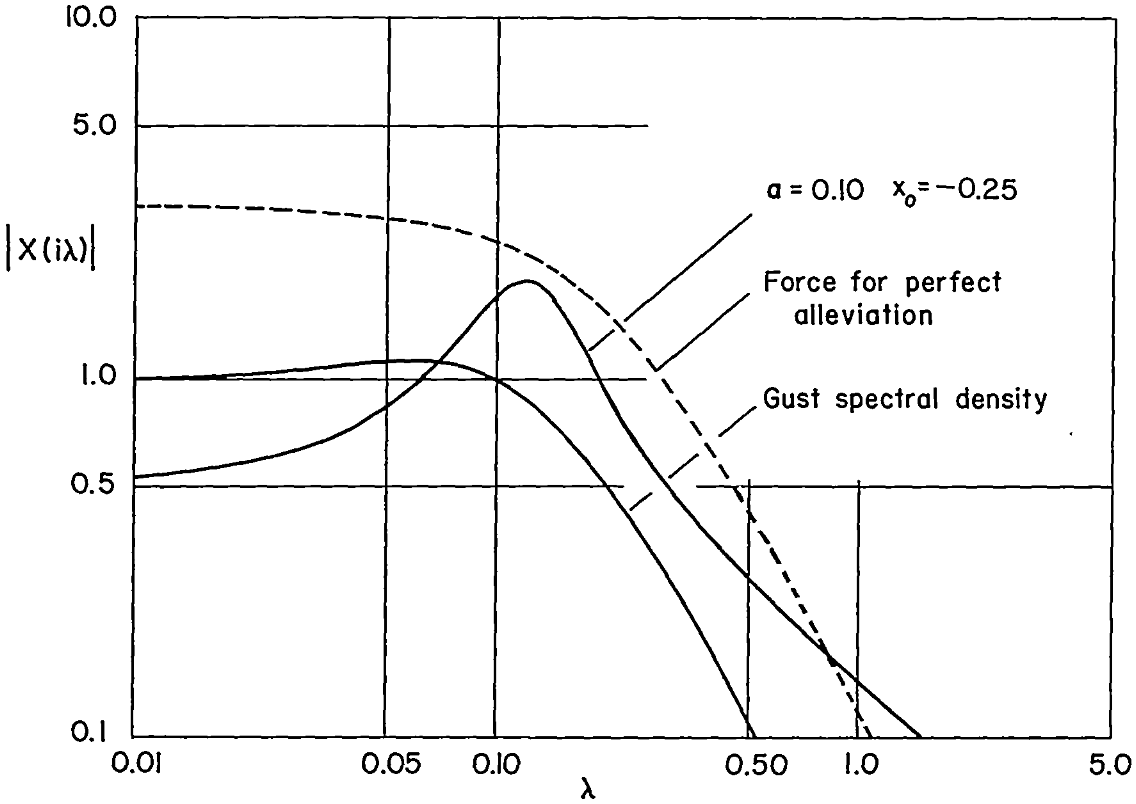


Figure 10.- Amplitude of control-force response to harmonic gusts.

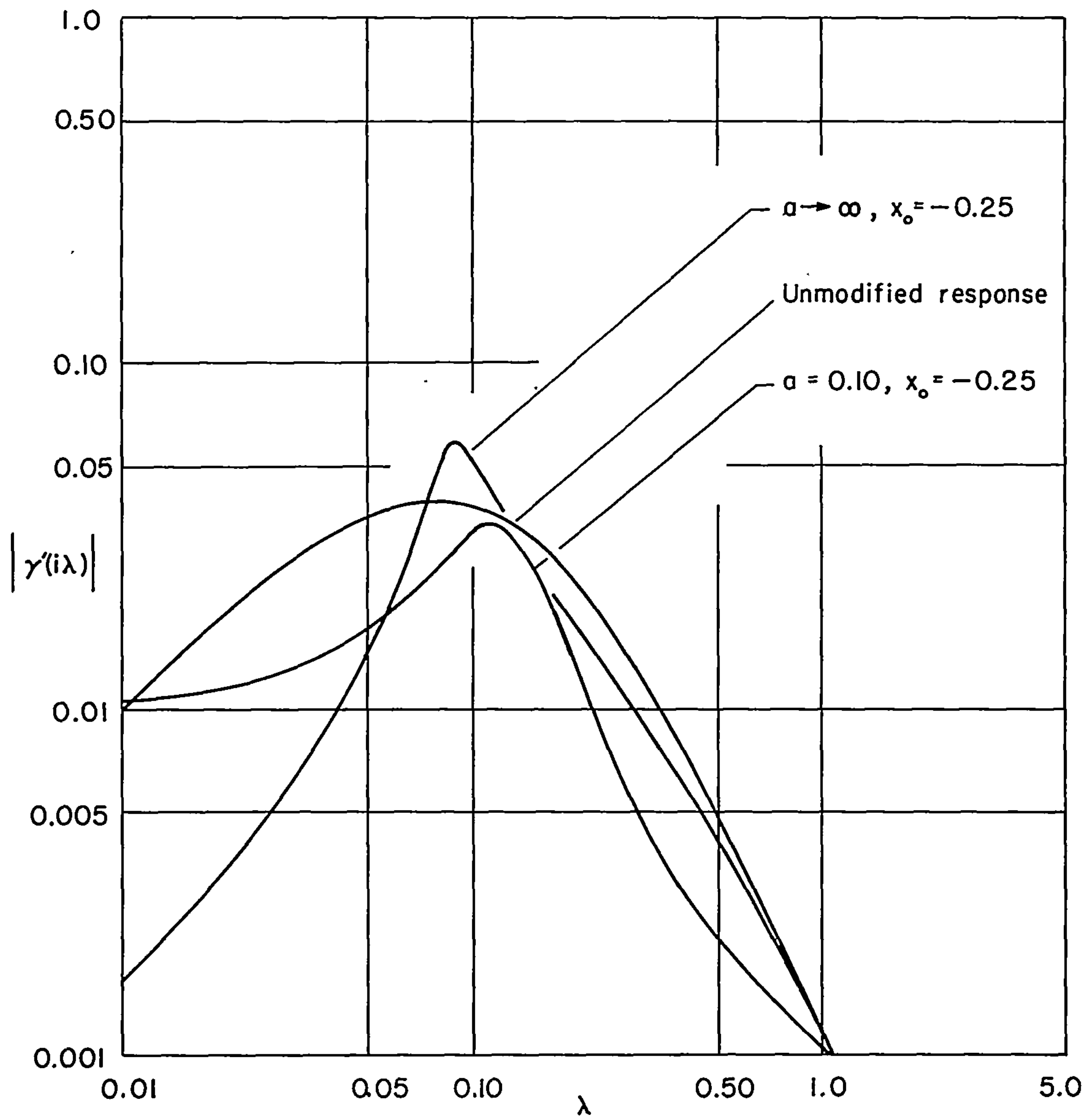


Figure 11.- Amplitude of normal acceleration response to harmonic gusts with and without operation of control force.

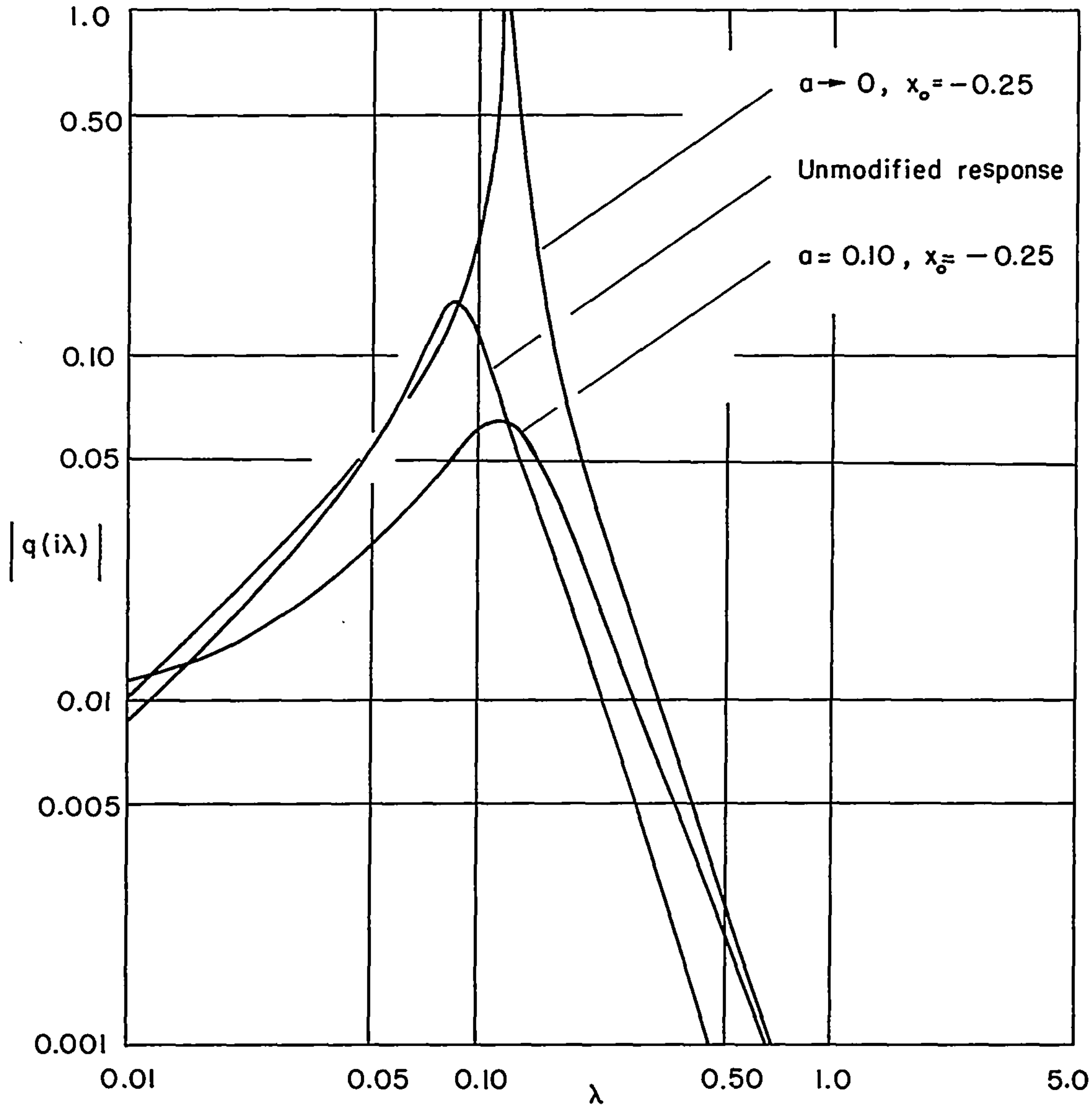


Figure 12.- Amplitude of pitching velocity response to harmonic gusts with and without operation of control force.

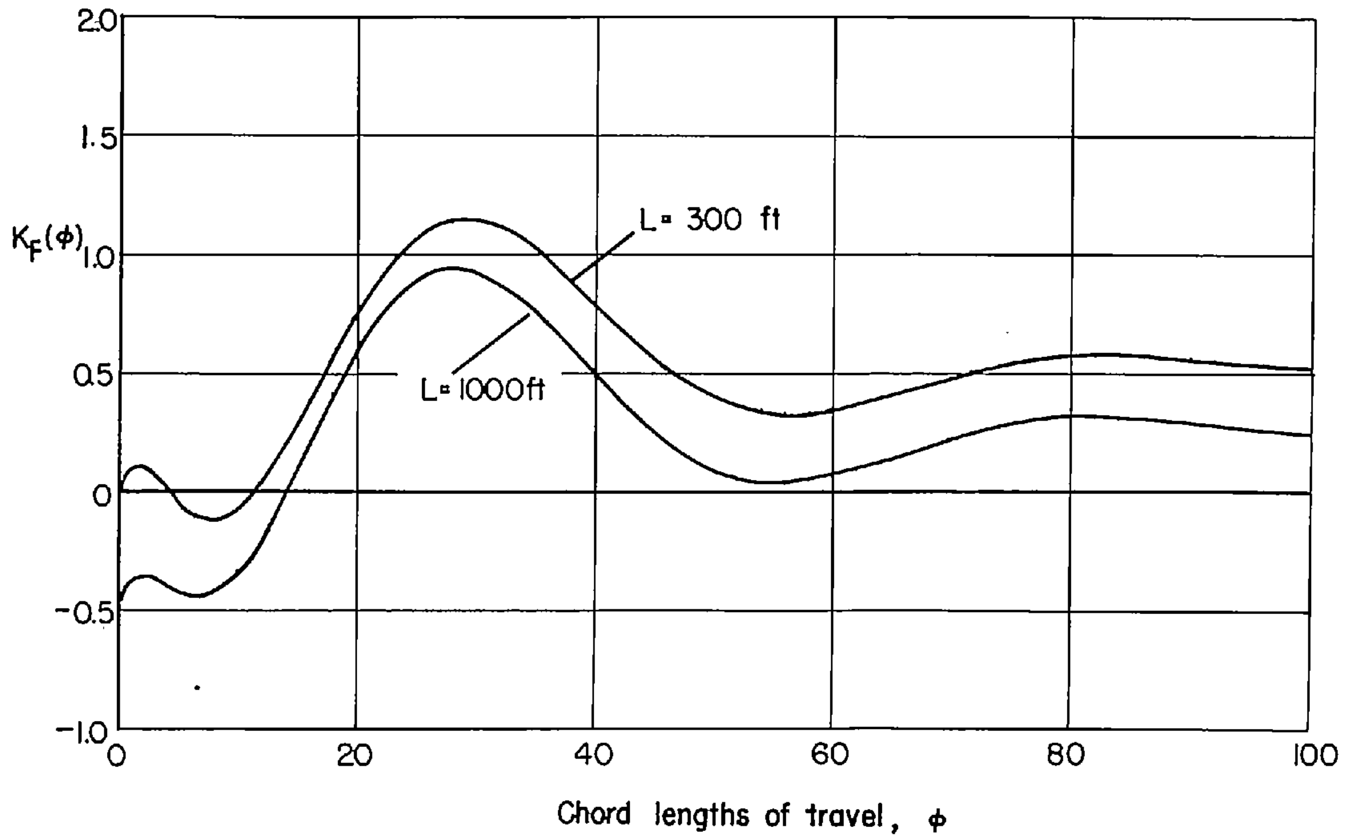


Figure 13.- Comparison of optimum control-force responses to sharp-edge gust for two values of scale of turbulence; $a = 0.10$, $x_o = -0.25$.

NACA - Langley Field, Va.